

Janossy densities for Unitary ensembles at the spectral edge

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Abstract

For a broad class of unitary ensembles of random matrices we demonstrate the universal nature of the Janossy densities of eigenvalues near the spectral edge, providing a different formulation of the probability distributions of the limiting second, third, etc. largest eigenvalues of the ensembles in question. The approach is based on a representation of the Janossy densities in terms of a system of orthogonal polynomials, plus the steepest descent method of Deift and Zhou for the asymptotic analysis of the associated Riemann-Hilbert problem.

1 Introduction

Consider the probability measure P_n on the space of $n \times n$ Hermitian matrices M defined by

$$dP_n(M) = \frac{1}{Z_n} e^{-n \operatorname{tr} V(M)} dM,$$

in which tr denotes the matrix trace, dM is the Lebesgue measure, and the potential V grows sufficiently fast at $\pm\infty$ so that the normalizer $Z_n < \infty$. This prescription is an instance of the *unitary ensembles* of Random Matrix Theory; the invariance $dP_n(U^* M U) = dP_n(M)$ for any $n \times n$ unitary matrix U explains the terminology.

Regarding their spectral properties these ensembles are integrable. That is to say, the joint probability density of the eigenvalues x_1, x_2, \dots, x_n induced by P_n may be computed:

$$\rho_n(x_1, \dots, x_n) = \frac{1}{\hat{Z}_n} \prod_{1 \leq \ell < k \leq n} |x_\ell - x_k|^2 e^{-n \sum_{k=1}^n V(x_k)}, \quad (1.1)$$

with a new normalizer \hat{Z}_n . Even more, all finite dimensional correlation functions of the eigenvalues,

$$\rho_n^{(k)}(x_1, \dots, x_k) \equiv \frac{n!}{(n-k)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \rho_n(x_1, \dots, x_k, \bar{x}_{k+1}, \dots, \bar{x}_n) d\bar{x}_{k+1} \cdots d\bar{x}_n, \quad (1.2)$$

have explicit expressions. Bring in the system of polynomials

$$p_{k,n}(x) = \gamma_{k,n} x^k + \dots,$$

$k = 1, \dots, n$ with $\gamma_{k,n} > 0$, orthonormal with respect to the weight $w_n(x) \equiv e^{-nV(x)}$ over \mathbb{R} . That is, $\int_{-\infty}^{\infty} p_{\ell,n}(x)p_{k,n}(x)w_n(x)dx = \delta_{\ell k}$, and it holds

$$\rho_n^{(k)}(x_1, \dots, x_k) = \det \left[K_n(x_\ell, x_m) \right]_{1 \leq \ell, m \leq k}, \quad (1.3)$$

in which

$$\begin{aligned} K_n(x, y) &= \sqrt{w_n(x)}\sqrt{w_n(y)} \sum_{k=0}^{n-1} p_{k,n}(x)p_{k,n}(y) \\ &= \sqrt{w_n(x)}\sqrt{w_n(y)} \frac{\gamma_{n-1,n}}{\gamma_{n,n}} \frac{p_{n,n}(x)p_{n-1,n}(y) - p_{n-1,n}(x)p_{n,n}(y)}{x - y}, \end{aligned} \quad (1.4)$$

by the formula of Christoffel-Darboux. The form of (1.3) implies that the ensemble eigenvalues comprise a *determinantal* point process.

With the above normalization, $\rho_n^{(k)}(x_1, \dots, x_k)$ is really a joint intensity of there being an eigenvalue, irrespective of order, at each of the points x_1 through x_k . Alternatively, fix a subset Γ of \mathbb{R} containing x_1, \dots, x_k . Then, the probability that there are exactly k eigenvalues in Γ , one at each of those same points, defines the k -th level Janossy density, denoted by $\mathcal{J}_{n,\Gamma}^{(k)}(x_1, \dots, x_k)$. For any determinantal point processes the Janossy densities are also determinantal ([5] p. 140): in our case,

$$\mathcal{J}_{n,\Gamma}^{(k)}(x_1, \dots, x_k) = D(\Gamma) \times \det \left[L_{n,\Gamma}(x_\ell, x_m) \right]_{1 \leq \ell, m \leq k}, \quad (1.5)$$

where

$$L_{n,\Gamma} = K_{n,\Gamma}(\mathbb{I} - K_{n,\Gamma})^{-1}, \quad (1.6)$$

the kernel $K_{n,\Gamma}(x, y)$ equaling $\mathbf{1}_\Gamma(x)K_n(x, y)\mathbf{1}_\Gamma(y)$, and the prefactor $D(\Gamma)$ is the Fredholm determinant

$$D(\Gamma) = \det(\mathbb{I} - K_{n,\Gamma}). \quad (1.7)$$

More important for what follows, it has recently been shown in [2] that kernel of $L_{n,\Gamma}$ is also Christoffel-Darboux type. In particular,

$$L_{n,\Gamma}(x, y) = \sqrt{w_n(x)}\sqrt{w_n(y)} \frac{\tilde{\gamma}_{n-1,n}}{\tilde{\gamma}_{n,n}} \frac{\tilde{p}_{n,n}(x)\tilde{p}_{n-1,n}(y) - \tilde{p}_{n-1,n}(x)\tilde{p}_{n,n}(y)}{x - y}, \quad (1.8)$$

where $\{\tilde{p}_{k,n}\}$ are the polynomials orthogonal to the weight $w_n(x)$, now restricted to the complement of Γ :

$$\int_{\mathbb{R} \setminus \Gamma} \tilde{p}_{\ell,n}(x)\tilde{p}_{k,n}(x)w_n(x)dx = \delta_{\ell k}.$$

For a large class of potentials V , [11] employs the Riemann-Hilbert Problem (*RHP*) characterization of the system $\{p_{k,n}\}$ to obtain sharp $n \rightarrow \infty$ asymptotics of the kernel K_n , and thus also the correlation functions $\rho_n^{(k)}$, in the bulk of the spectrum. Here we take up the analogous project for the Janossy densities at the spectral edge by analyzing the $\{\tilde{p}_{k,n}\}$ system.

The requirements on the potential V are described in terms of the equilibrium measure μ_V , or weak limit of the eigenvalue counting measure. This may be characterized as the infimum of

$$I_V(\mu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int_{-\infty}^{\infty} V(x) d\mu(x), \quad (1.9)$$

over the space of probability measures on \mathbb{R} . Now, if

$$V : \mathbb{R} \rightarrow \mathbb{R} \text{ is real analytic} \quad (1.10)$$

and

$$\lim_{|x| \rightarrow \infty} \frac{V(x)}{\log(x^2 + 1)} = +\infty, \quad (1.11)$$

then [7] proves that this infimum is uniquely attained at μ_V . Further, μ_V possesses a density $\psi_V(x)$ with compact support comprised of a finite number of intervals. Assumptions (1.10) and (1.11) are adopted here. By a scaling we may fix the rightmost edge of the support of $\psi_V(x)$ at $x = 1$, and we further assume that

$$\psi_V(x) \text{ is regular.} \quad (1.12)$$

By this we will mean the following.

- (a) ψ_V vanishes like a square-root at each endpoint of $\text{supp}(\mu_V)$.
- (b) ψ_V is strictly positive in the interior of $\text{supp}(\mu_V)$.
- (c) Strict inequality holds in the characterizing Euler-Lagrange equations in the exterior of $\text{supp}(\mu_V)$, see (2.10).

One imagines that square-root vanishing at $x = 1$ would suffice; full regularity has been assumed for technical reasons.

For $V(x)$ satisfying (1.10), (1.11) and (1.12), one may infer from the results in [11] that the kernel K_N at the spectral edge has the universal limit,

$$\lim_{n \rightarrow \infty} \frac{1}{c_V n^{2/3}} K_n \left(1 + \frac{x}{c_V n^{2/3}}, 1 + \frac{y}{c_V n^{2/3}} \right) = \frac{\text{Ai}(x) \text{Ai}'(y) - \text{Ai}'(x) \text{Ai}(y)}{x - y}, \quad (1.13)$$

with constant $c_V > 0$ and $\text{Ai}(\cdot)$ the Airy function.¹ Based on this, it is expected that the kernel $L_{n,\Gamma}$ for $\Gamma = [1 + \alpha/(c_V n^{2/3}), \infty)$ with any real α will have a universal limit as $n \rightarrow \infty$. We introduce the shorthand,

$$L_{n,\alpha}(x, y) \equiv L_{n, [1 + \alpha/(c_V n^{2/3}), \infty)}(x, y), \quad (1.14)$$

and, noting that the regime of interest is for $x, y \in \Gamma$, prove the following.

Theorem 1.1. *Assume that the potential $V(x)$ satisfies (1.10), (1.11) and (1.12). Then, there are pairs of functions $\{f_\alpha^\rightarrow(z), g_\alpha^\rightarrow(z)\}$ and $\{f_\alpha^\leftarrow(z), g_\alpha^\leftarrow(z)\}$ defined for $\alpha > 0$ and $\alpha \leq 0$ respectively, such that the following universal asymptotics hold. For $\alpha > 0$,*

$$\frac{1}{c_V n^{2/3}} L_{n,\alpha} \left(1 + \frac{x}{c_V n^{2/3}}, 1 + \frac{y}{c_V n^{2/3}} \right) = \frac{f_\alpha^\rightarrow(x) g_\alpha^\rightarrow(y) - g_\alpha^\rightarrow(x) f_\alpha^\rightarrow(y)}{x - y} + \mathcal{O}(n^{-2/3}), \quad (1.15)$$

¹Though understood to hold in greater generality, a detailed proof of (1.13) actually only appears in the literature for polynomial V [10].

while for $\alpha \leq 0$,

$$\frac{1}{c_V n^{2/3}} L_{n,\alpha} \left(1 + \frac{x}{c_V n^{2/3}}, 1 + \frac{y}{c_V n^{2/3}} \right) = \frac{f_\alpha^-(x-\alpha)g_\alpha^-(y-\alpha) - g_\alpha^-(x-\alpha)f_\alpha^-(y-\alpha)}{x-y} + \mathcal{O}(n^{-2/3}). \quad (1.16)$$

Both estimates are uniform for x and y restricted to compact sets of (α, ∞) .

The functions $f_\alpha^\pm(z)$ and $g_\alpha^\pm(z)$ are read off from the solutions of a pair of 2×2 RHPs denoted by RHP^\rightarrow for $\alpha > 0$ and RHP^\leftarrow for $\alpha < 0$. With the corresponding contours and their orientations depicted in Figure 1, we have:

RHP^\rightarrow ($\alpha > 0$): Seek a 2×2 matrix valued function $M^\rightarrow(z)$, analytic in $\mathbb{C} \setminus \Sigma^\rightarrow$ such that:

$$\begin{aligned} (M^\rightarrow)_+(z) &= (M^\rightarrow)_-(z) \begin{pmatrix} 1 & e^{-\frac{4}{3}z^{3/2}} \\ 0 & 1 \end{pmatrix}, \quad z \in (0, \alpha), \\ (M^\rightarrow)_+(z) &= (M^\rightarrow)_-(z) \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}z^{3/2}} & 1 \end{pmatrix}, \quad \arg z = \pm \frac{2}{3}\pi, \\ (M^\rightarrow)_+(z) &= (M^\rightarrow)_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-\infty, 0) \end{aligned} \quad (1.17)$$

with

$$M^\rightarrow(z) = z^{-\frac{1}{4}\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{\pi i}{4}\sigma_3} (I + \mathcal{O}(z^{-1})), \quad z \rightarrow \infty. \quad (1.18)$$

RHP^\leftarrow ($\alpha < 0$): Now seek a 2×2 matrix valued function $M^\leftarrow(z)$, analytic in $\mathbb{C} \setminus \Sigma^\leftarrow$ such that:

$$\begin{aligned} (M^\leftarrow)_+(z) &= (M^\leftarrow)_-(z) \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}z^{3/2} + 2\alpha z^{1/2}} & 1 \end{pmatrix}, \quad \arg z = \pm \frac{2}{3}\pi, \\ (M^\leftarrow)_+(z) &= (M^\leftarrow)_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-\infty, 0), \end{aligned} \quad (1.19)$$

with the same asymptotics as $z \rightarrow \infty$.

Note that the problems coincide at $\alpha = 0$. In either problem, $(M^\pm)_\pm(z)$ indicate the limits of $M^\pm(z)$ as z approaches either Σ^\rightarrow or Σ^\leftarrow from the positive or negative sides (the precise sense in which the limit holds is discussed later). Last, σ_3 denotes the third Pauli matrix, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

A large part of this paper is dedicated to the proof that there exist unique solutions to RHP^\rightarrow and RHP^\leftarrow . Granting that, we may now define the functions comprising the limiting kernels (1.15) and (1.16), hereafter denoted $\mathbb{A}_\alpha(x, y)$ and $\mathbb{B}_\alpha(x, y)$.

Definition 1.2. For $-\frac{2}{3}\pi < \arg z < \frac{2}{3}\pi$ and $z \notin [0, \alpha]$,

$$(f_\alpha^\rightarrow(z), g_\alpha^\rightarrow(z)) = \frac{1}{\sqrt{2\pi}} e^{\frac{\pi i}{4}} e^{-\frac{2}{3}z^{3/2}} ((M^\rightarrow)_{11}(z), (M^\rightarrow)_{21}(z)). \quad (1.20)$$

Similarly,

$$(f_\alpha^\leftarrow(z), g_\alpha^\leftarrow(z)) = \frac{1}{\sqrt{2\pi}} e^{\frac{\pi i}{4}} e^{-(\frac{2}{3}z^{3/2} + \alpha z^{1/2})} ((M^\leftarrow)_{11}(z), (M^\leftarrow)_{21}(z)) \quad (1.21)$$

for all z with $-\frac{2}{3}\pi < \arg z < \frac{2}{3}\pi$.

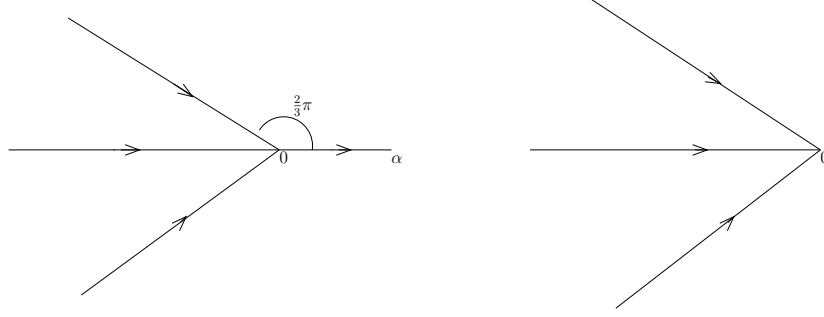


Figure 1: The contours Σ^{\rightarrow} and Σ^{\leftarrow} for RHP^{\rightarrow} and RHP^{\leftarrow}

One concludes that $f_{\alpha}^{\rightarrow}(x)$ and $g_{\alpha}^{\rightarrow}(x)$ are real analytic for $x > \alpha$ and $x > 0$; the diagonals $\mathbb{A}_{\alpha}(x, x)$ and $\mathbb{B}_{\alpha}(x, x)$ for $x > \alpha$ are therefore well defined. As for their behavior as functions of α :

Theorem 1.3. *Both $\mathbb{A}_{\alpha}(x, y)$ and $\mathbb{B}_{\alpha}(x, y)$ are continuous functions of α for fixed x, y . Continuity holds down (or up) to $\alpha = 0$ from either side.*

Finally, while we have not expressed the limit kernel in terms of known special functions, we do have the following asymptotics.

Theorem 1.4. *Uniformly for z in compact sets of $(0, \infty)$,*

$$f_{\alpha}^{\rightarrow}(z) = \text{Ai}(z) \left(1 + \mathcal{O}(e^{-\alpha^{3/2}})\right), \quad g_{\alpha}^{\rightarrow}(z) = \text{Ai}(z) \left(1 + \mathcal{O}(e^{-\alpha^{3/2}})\right) \quad (1.22)$$

as $\alpha \rightarrow +\infty$, while

$$\begin{aligned} f_{\alpha}^{\leftarrow}(z) &= (|\alpha| - \frac{2}{3}z)^{1/2} I_0(z^{1/2}(|\alpha| - \frac{2}{3}z)) \left(1 + \mathcal{O}(|\alpha|^{-1})\right), \\ g_{\alpha}^{\leftarrow}(z) &= -2\pi I_0'(z^{1/2}(|\alpha| - \frac{2}{3}z)) \left(1 + \mathcal{O}(|\alpha|^{-1})\right) \end{aligned} \quad (1.23)$$

as $\alpha \rightarrow -\infty$. $I_0(\cdot)$ is the modified Bessel function of the first kind.

After describing applications of Theorem 1.1 to the limiting distributions of the largest eigenvalues for Unitary ensembles and some possible extensions, the analysis begins in Section 2 where the RHP connected to the polynomials $\{\tilde{p}_{k,n}\}$ is introduced. Section 3 subjects this RHP to a series of transformations, following the Deift-Zhou method of steepest descent [6]. A local analysis for the problem in the vicinity of $z = 1$ in terms of RHP^{\leftarrow} and RHP^{\rightarrow} is detailed in Section 4. With these parametrices, Theorem 1.1 is proved in Section 5. Section 6 is devoted to the existence question for RHP^{\leftarrow} and RHP^{\rightarrow} ; this is accomplished by a general vanishing lemma argument. Section 7 establishes several properties of those solutions, including their continuity and asymptotics (Theorems 1.3 and 1.4).

Remark The results here should be compared with those in the recent paper [4] which considers the following set-up: Take V regular with right-most edge of μ_V placed at the origin, and seek the asymptotics of the corresponding orthogonal polynomial kernel for the weight $e^{-nV(x)}$ restricted to

$(-\infty, 0]$. This is the same starting point as our problem. In [4] though, a parameter $c = c(n) > 0$ is introduced in the weight as in cV which, when adjusted, can move the edge of the support of μ_V away from the origin or push it in, creating a “hard-edge”, or square-root singularity at the origin. While different at all finite n , this device is qualitatively the same as our choice of the sign of α , and should lead to the same phase transition in the limit. By a quadratic transformation, the authors of [4] are able to write the limiting kernel for points x and y to the left of the origin in terms of Painlevé II. For the probabilistic motivations here, it is the kernel to the right of the critical point which is important. By analogy this should correspond to the kernel in [4] along the imaginary axis; the precise relationship remains to be worked out.

1.1 Janossy densities and the distribution of the largest eigenvalues

Denote the ordered eigenvalues of M by $\lambda_1 > \lambda_2 > \dots$. The well known gap formula for determinantal ensembles with kernel K_n states that: for any $B \subset \mathbb{R}$,

$$P(\text{there are exactly } m \text{ eigenvalues in } B) = \frac{-1^m}{m!} \frac{d^m}{d\theta^m} \det(I - \theta K_n \mathbf{1}_B) \Big|_{\theta=1}. \quad (1.24)$$

With $m = 0$, this formula together with the limiting result (1.13) implies

$$\lim_{n \rightarrow \infty} P\left(\lambda_1 \leq 1 + \frac{\alpha}{c_V n^{2/3}}\right) = \det(\mathbb{I} - K_{\text{Airy}} \mathbf{1}_{[\alpha, \infty)}), \quad (1.25)$$

with K_{Airy} standing in for the L^2 -operator with Airy kernel, see again [10] for a full proof in the case of polynomial V . The celebrated result of Tracy and Widom ([20], with extensions in [21]) provides a closed form for this Fredholm determinant, to wit,

$$\det(\mathbb{I} - K_{\text{Airy}} \mathbf{1}_{[\alpha, \infty)}) = \exp\left(-\int_{\alpha}^{\infty} (s - \alpha) u^2(s) ds\right) \equiv F_{TW}(\alpha), \quad (1.26)$$

in which $u(s)$ is the unique solution of Painlevé II with $u(s) \sim \text{Ai}(s)$ as $s \rightarrow +\infty$.

Formulas for the limiting distributions of the scaled λ_2, λ_3 , etc, also exist. This is also found in [20], though note there the asymptotics are only taken on for GUE. To explain, first replace the appearance of $u(s)$ in (1.26) with $u(s; \theta)$ determined by the same equation but with $u(s; \theta) \sim \sqrt{\theta} \text{Ai}(s)$ at infinity, and denote the corresponding exponential function $F(\alpha; \theta)$. Then, following (1.24), $\frac{-1^m}{m!} \times \partial_{\theta}^{(n)} F(\alpha; \theta)$ evaluated at $\theta = 1$ yields the (limiting) probability of there being exactly m eigenvalues larger than α . The corresponding distribution functions can then be built in the obvious manner.

The Janossy densities provide a different path to the law of the scaled largest eigenvalues. From the definition (1.5), we have that

$$\begin{aligned} P(\text{exactly } m \text{ eigenvalues in } B) &= P(\text{no eigenvalues in } B) \\ &\quad \times \frac{1}{m!} \int_B \cdots \int_B \det[L_{n,B}(x_{\ell}, x_k)]_{1 \leq \ell, k \leq m} dx_1 \cdots dx_m, \end{aligned}$$

and, assuming (1.25), one can obtain the following from Theorem 1.1.

Corollary 1.5. *With \mathbb{M}_α equal to \mathbb{A}_α for $\alpha > 0$ and \mathbb{B}_α for $\alpha < 0$,*

$$\lim_{n \rightarrow \infty} P\left(\lambda_m \leq 1 + \frac{\alpha}{c_V n^{2/3}}\right) = F_{TW}(\alpha) \times \sum_{n=0}^{m-1} \frac{1}{n!} \int_{\alpha}^{\infty} \cdots \int_{\alpha}^{\infty} \det\left(\mathbb{M}_\alpha(x_\ell, x_k)\right) dx_1 \cdots dx_n. \quad (1.27)$$

This describes the general limit distribution as that of the largest eigenvalue modulated by a finite sum of (standard) determinants with a universal kernel. Again, at this point the kernel is only defined in terms of a pair of *RHPs*, and so the above form of limit law is far from optimal.

Proof of Corollary 1.5. We provide just a sketch, using estimates developed below. To produce the second factor in (1.27), one must pass the point-wise convergence of the kernel established Theorem 1.1 under the integral. The fast decay of the exponential weight will control the integral at infinity, while more information is needed to deal with the integral near α . In particular, here one wants to show that $n^{-2/3} L_{n,\alpha}(1 + n^{-2/3}x, 1 + n^{-2/3}x)$ is uniformly integrable over $x \in [\alpha, \alpha + \epsilon]$. Local forms of the solution of *RHP* $^{\vec{\tau}}$, provided in (6.8) for $\alpha > 0$ and (6.11)-(6.13) for $\alpha < 0$, show that the first columns of $M^{\vec{\tau}}$ along with their derivatives are bounded down to α or the origin (from the right). By Definition 1.2 we then see that, even on diagonal, the limit kernel ($\mathbb{M}_\alpha(x, x) = f'_\alpha(x)g_\alpha(x) - f_\alpha(x)g'_\alpha(x)$, neglecting the $(\vec{\tau})$ -superscripts) is integrable near α . Next, (5.3) and (5.5) express the finite n kernel in terms of certain (well-behaved) auxiliary functions and the first column of $M^{\vec{\tau}}(z)$. A simple analysis of those auxiliary functions shows that $n^{-2/3} L_{n,\alpha}$ inherits the integrability of $M^{\vec{\tau}}$ and yields the result. \square

Improved asymptotics for the Janossy kernel might also provide estimates on the speed of convergence to the Tracy-Widom law (analogues of either the Berry-Esseen estimates or Edgeworth expansions for the classical central limit theorem). Results of this type are important in multivariate statistics, and have already been established for GUE and the related LUE in [14] and [3]. The case of unitary ensembles with non-quadratic potentials has not been explored. Note however from the general formula (1.6) we have

$$\begin{aligned} \frac{d}{d\alpha} \log P\left(\lambda_1 \leq 1 + \frac{\alpha}{c_V n^{2/3}}\right) &= \frac{d}{d\alpha} \log \det\left(\mathbb{I} - K_n \mathbf{1}_{[1+\alpha c_V^{-1} n^{-2/3}, \infty)}\right) \\ &= \frac{1}{c_V n^{2/3}} L_{n,\alpha}\left(1 + \frac{\alpha}{c_V n^{2/3}}, 1 + \frac{\alpha}{c_V n^{2/3}}\right). \end{aligned} \quad (1.28)$$

A similar expression at $n = \infty$ is a first step in the derivation of (1.26), and the limiting kernels (1.15) and (1.16) are not surprisingly tied to the resolvent kernel of the Airy operator, see [9] and [22]. More to the point, a suitable expansion in n in (1.28) would bound the convergence speed. From Theorem 1.1 one anticipates the rate is $n^{-2/3}$, and that is just what is proven for GUE and LUE. Of course, carrying out the suggested program requires sharp asymptotics of $L_{n,\alpha}(x, y)$ along the diagonal ($x = y = \alpha$). An estimate of the form

$$\left| \frac{1}{c_V n^{2/3}} L_{n,\alpha}\left(1 + \frac{\alpha}{c_V n^{2/3}}, 1 + \frac{\alpha}{c_V n^{2/3}}\right) - \mathbb{M}_\alpha(\alpha, \alpha) \right| \leq n^{-2/3} \phi(\alpha)$$

with $\phi(\alpha)$ integrable at positive infinity would, for example, more than suffice.

2 First RHP and introduction to the calculation

The starting point is the *RHP* characterization of orthogonal polynomials due to Fokas, Its and Kitaev [13]. Fix a half-line $\Gamma = (-\infty, c]$ and consider the polynomials

$$\{\tilde{p}_{k,n} = \tilde{\gamma}_{k,n}x^k + \dots\} \text{ orthonormal with respect to } w_n(x) = e^{-nV(x)} \text{ for } x \in \Gamma. \quad (2.1)$$

Then, the *RHP* reads as follows.

RHP for Y : Seek a 2×2 matrix valued function $Y(z) = Y_n(z)$ such that

$$\begin{aligned} Y(z) & \text{ analytic in } \mathbb{C} \setminus \Gamma, \\ Y_+(z) &= Y_-(z) \begin{pmatrix} 1 & w_n(z) \\ 0 & 1 \end{pmatrix}, \quad z \in \Gamma, \\ Y(z) &= \left(I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z \rightarrow \infty. \end{aligned} \quad (2.2)$$

The second, or “jump”, condition, is read as

$$Y_{\pm}(z) \equiv \lim_{z' \rightarrow z} Y(\zeta), \quad z \in \Gamma, \quad z' \in \mathbb{C}_{\pm}, \text{ the upper or lower half-plane.} \quad (2.3)$$

This can be understood in the sense of continuous boundary values for $z \in \Gamma$ away from the endpoint $z = c$ with the additional condition that

$$Y(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log |z - c|) \\ \mathcal{O}(1) & \mathcal{O}(\log |z - c|) \end{pmatrix}, \quad z \text{ near } c.$$

With that said, the basic result is that the unique solution of this *RHP* is given by

$$Y(z) = \begin{pmatrix} \frac{1}{\tilde{\gamma}_{n,n}} \tilde{p}_{n,n}(z) & \frac{1}{\tilde{\gamma}_{n,n}} C(\tilde{p}_{n,n} w_n)(z) \\ -2\pi i \tilde{\gamma}_{n-1,n} \tilde{p}_{n-1,n}(z) & -2\pi i \tilde{\gamma}_{n-1,n} C(\tilde{p}_{n-1,n} w_n)(z) \end{pmatrix}, \quad (2.4)$$

where C denotes the Cauchy operator on Γ :

$$Cf(z) = C_{\Sigma}f(z) \equiv \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s - z} ds, \quad z \notin \Sigma,$$

for any contour $\Sigma \subset \mathbb{C}$ and function $f(z) \in L^2(\Sigma, |dz|)$.

Note that (2.4) contains the (w_n, Γ) orthogonal polynomials of degrees $n-1$ and n in its first column. It follows that the kernel of interest, $L_{n,\alpha}$, may be expressed entirely in terms $(Y_{11}(z), Y_{21}(z))$ where Γ is now a function of both n and α : we have in particular,

$$\Gamma = \Gamma_{n,\alpha} = \left(-\infty, 1 + \alpha c_V^{-1} n^{-2/3}\right], \quad (2.5)$$

and will use the additional shorthand $c_{n,\alpha} \equiv \alpha c_V^{-1} n^{-2/3}$ for the (moving) endpoint.

The analysis of Y for $n \rightarrow \infty$ entails a series of transformations, $Y \mapsto T \mapsto S \mapsto R$, in order to obtain a *RHP* for R which is normalized at infinity (*i.e.*, $R(z) \rightarrow I$ as $z \rightarrow \infty$), and has jump

matrices which are uniformly close to the identity as $n \rightarrow \infty$. Afterwards, unfolding this series of transformations will produce the asymptotics of Y . We are primarily concerned with the behavior of $Y(z)$ in the vicinity of $z = 1$, for which we will build local parametrices. The basic program is identical to that in the analysis of the *RHP* connected to orthogonal polynomials over the full line in [11] or [12]. Novel here is that the problem will follow two different paths, depending on the sign of α .

2.1 Equilibrium measures and the g -function

The first transformation, $Y \mapsto T$, rests on properties of the density which minimizes the analogue of I_V in (1.9). We begin by recalling several properties of ψ_V , the “unconstrained” equilibrium density connected to the analysis of the full-line orthogonal polynomials.

As indicated in the introduction, the support of ψ_V is a union of $(N + 1)$ disjoint intervals and we normalize the right endpoint to sit at 1. The intervals of support are referred to as the *bands*; the complementary N intervals making up the *gaps*. Following [11], the interior of the support is denoted by

$$J = \bigcup_{k=1}^{N+1} (b_{k-1}, a_k),$$

and the density ψ_V can be written,

$$\psi_V(z) = \frac{1}{2\pi i} R_+^{1/2}(z) h_V(z), \quad \text{for } z \in J, \quad (2.6)$$

in which

$$R(z) = \prod_{k=1}^{N+1} (z - b_{k-1})(z - a_k), \quad (2.7)$$

and h_V is real analytic on \mathbb{R} . Here, the branch of $R(z)$ is chosen so that $R(z)$ behaves like z^{N+1} as $z \rightarrow \infty$.

With this, the first transformation of the *RHP* for the unconstrained polynomials is based on the introduction of the g -function,

$$g(z) = \int_{\mathbb{R}} \log(z - x) \psi_V(x) dx. \quad (2.8)$$

The function $g(z)$ is analytic on $\mathbb{C} \setminus (-\infty, 1]$ and has the following properties. First, there a constant ℓ so that

$$g_+(z) + g_-(z) - V(z) - \ell = 0, \quad \text{for } z \in \bar{J}, \quad (2.9)$$

$$g_+(z) + g_-(z) - V(z) - \ell < 0, \quad \text{for } z \in \mathbb{R} \setminus \bar{J}. \quad (2.10)$$

(The strict inequality in (2.10) is our final regularity condition.) Second, it holds that

$$g_+(z) - g_-(z) = 2\pi i \int_z^1 \psi_V(x) dx, \quad \text{for } z \in (-\infty, 1). \quad (2.11)$$

That is, $g_+(z) - g_-(z)$ is purely imaginary on \mathbb{R} and constant in each of the gaps, with the more detailed picture being:

$$g_+(z) - g_-(z) = \begin{cases} 2\pi i, & z \in (-\infty, b_0), \\ 2\pi i \int_{b_k}^1 \psi_V(x) dx \equiv 2\pi i \Omega_k, & z \in (a_k, b_k), \\ 0, & z \in (1, +\infty). \end{cases} \quad (2.12)$$

Appraisals (2.9) through (2.12) are basic consequences of the Euler-Lagrange equations for (1.9), as is explained in Section 3.2 of [11].

With $\{\tilde{p}_{k,n}\}$ we are working on the n -dependent interval $\Gamma_{n,\alpha}$, but may proceed in a like manner. For each integer n and real α , the old reasoning will show that infimum of

$$I_{n,\alpha}(\mu) = \int_{\Gamma_{n,\alpha}} \int_{\Gamma_{n,\alpha}} \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int_{\Gamma_{n,\alpha}} V(x) d\mu(x) \quad (2.13)$$

is uniquely achieved. The minimizing density is however qualitatively different $\alpha > 0$ or $\alpha < 0$, and is denoted by ψ_V^\rightarrow or ψ_V^\leftarrow respectively.

2.1.1 The case $\alpha > 0$

When $\alpha > 0$, $\Gamma_{n,\alpha}$ contains the support of the full-line minimizer ψ_V , and we state without proof the following.

Lemma 2.1. *It holds that $\psi_V^\rightarrow(z) = \psi_V(z)$. Thus, by assumption (1.12),*

$$\psi_V^\rightarrow(x) = (1-x)^{1/2} \beta_V(x), \quad x \in (1-\delta, 1), \quad (2.14)$$

for all small $\delta > 0$. Here, $\beta_V(z)$ is analytic in a neighborhood of $z = 1$, $\beta_V(1) > 0$, and for later we remark $c_V \equiv (\beta_V(1)/2)^{2/3}$.

Note that a definition of $\beta_V(z)$ is implicit in (2.6). Also, (2.14) has been set apart as this regime is of central importance in what follows.

2.1.2 The case $\alpha < 0$

For $\alpha < 0$, one attains different minima in (2.13) and (1.9). Still, using the assumed regularity we will show that, for all n large enough, the support of ψ_V^\leftarrow consists of $N+1$ intervals,

$$J_{n,\alpha} = \bigcup_{k=1}^{N+1} (b_{k-1}(n, \alpha), a_k(n, \alpha)),$$

with a_{N+1} fixed at $c_{n,\alpha} = 1 + \alpha/(c_V n^{2/3})$. Now setting,

$$\tilde{R}_{n,\alpha}(z) = \frac{(z - b_N)}{(z - a_{N+1})} \prod_{k=1}^N (z - b_{k-1})(z - a_k), \quad (2.15)$$

(from here on we suppress the (n, α) -dependence of the endpoints) we have:

Lemma 2.2. For $\alpha < 0$ and V satisfying (1.12) it holds,

$$\psi_V^\leftarrow(z) = \frac{1}{2\pi i} (\tilde{R}_{n,\alpha}(z))_+^{1/2} \left[(c_{n,\alpha} - z) h_{V,n,\alpha}(z) + C_{n,\alpha} \right], \quad (2.16)$$

on its support, with

$$C_{n,\alpha} = \frac{1}{2} \left(\frac{\alpha}{c_V n^{2/3}} \right) h_V(1) + \mathcal{O}(n^{-4/3}),$$

where h_V is as in (2.6) and $h_{V,n,\alpha}(z)$ is real analytic and tends to h_V as $n \rightarrow \infty$. The analogue of (2.14) reads

$$\psi_V^\leftarrow(x) = (c_{n,\alpha} - x)^{1/2} \beta_{V,n,\alpha}^{(a)}(x) + \frac{1}{2} \left(\frac{\alpha}{c_V n^{2/3}} \right) (c_{n,\alpha} - x)^{-1/2} \beta_{V,n,\alpha}^{(b)}(x), \quad x \in [1 - \delta, c_{n,\alpha}], \quad (2.17)$$

where $\beta_{V,n,\alpha}^{(a,b)}(z)$ are analytic in a neighborhood of $z = 1$ and $\lim_{n \rightarrow \infty} \beta_{V,n,\alpha}^{(a,b)}(c_{n,\alpha}) = \beta_V(1)$.

With ψ_V^\leftarrow in hand, we define a g -function exactly as in (2.8). The basic relations (2.9) through (2.10) remain valid, though with adjusted values for ℓ and the Ω 's.

Example (GUE). It is instructive to first spell out the computation for GUE , or when $V(z) = 2z^2$. The unrestricted minimizer has one band of support, $[-1, 1]$, and is given by the semi-circle law, $\psi_{GUE}(x) = \frac{2}{\pi} \sqrt{1 - x^2}$. Now fixing the right edge at $a = 1 - \varepsilon$ for small $\varepsilon > 0$, there is still one band $[b, a]$ with b and the adjusted density ψ_{GUE}^ε to be identified. If we put

$$G(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{\psi_{GUE}^\varepsilon(s)}{s - z} ds,$$

differentiating the relation (2.9) produces the scalar RHP :

$$G_+(z) + G_-(z) = \frac{4i}{\pi} z, \quad z \in [b, a], \quad \text{and} \quad G_+(z) - G_-(z) = 0, \quad z \in \mathbb{R} \setminus [b, a]. \quad (2.18)$$

Introduce $\tilde{R}(z) = \tilde{R}_\varepsilon(z) \equiv \frac{z-b}{z-a}$ and multiply both sides of (2.18) through by the square-root of this object. The RHP is then transformed into a standard form, and one finds

$$G(z) = \frac{\sqrt{\tilde{R}(z)}}{2\pi i} \int_b^a \frac{(4is/\pi)}{(\sqrt{\tilde{R}(s)})_+} \frac{ds}{s - z}, \quad (2.19)$$

subject to the single moment condition,

$$\frac{\pi}{2i} = \int_b^a s \sqrt{\frac{a-s}{s-b}} ds, \quad (2.20)$$

which holds since $zG(z) \rightarrow \frac{1}{\pi i}$ as $z \rightarrow \infty$. The integral (2.20) is easily computed, and $b = b(a) = \frac{1}{3}(a - 2\sqrt{a^2 + 3})$ for positive $a \leq 1$. Also, by properties of the Stieltjes transform, it holds that

$$\psi_{GUE}^\varepsilon(x) = \text{Re}(G_+(x)) = \frac{2}{\pi} \sqrt{\frac{x-b}{a-x}} \left(\frac{a-b}{2} - x \right), \quad x \in (a, b).$$

Now, for ε small and x near a ,

$$\psi_{GUE}^\varepsilon(x) = \left((1 - \varepsilon - x)^{1/2} + \frac{\varepsilon}{2} (1 - \varepsilon - x)^{-1/2} \right) (2\sqrt{2}/\pi) (1 + \mathcal{O}(\varepsilon)), \quad (2.21)$$

after substituting $a = 1 - \varepsilon$, $b = -1 + \mathcal{O}(\varepsilon^2)$ in the previous display. This provides a model for the general formula (2.17). \square

Proof of Lemma 2.2. As in the previous example, set $\varepsilon = \alpha c_V^{-1} n^{-2/3}$ with $a_{N+1} = a_{N+1}(\varepsilon) = 1 - \varepsilon$. Following the standard approach, the density is given by

$$\psi_{V,\varepsilon}^{\leftarrow}(z) = \operatorname{Re} \left((G_\varepsilon)_+(z) \right),$$

where,

$$G_\varepsilon(z) = \frac{1}{2\pi i} \frac{\sqrt{R_\varepsilon(z)}}{(z - a_{N+1}(\varepsilon))} \int_{J_\varepsilon} \frac{iV'(s)/\pi}{(\sqrt{R_\varepsilon(s)})_+} (a_{N+1}(\varepsilon) - s) \frac{ds}{s - z}, \quad (2.22)$$

and

$$R_\varepsilon(z) = \prod_{\ell=1}^{N+1} (z - b_{k-1}(\varepsilon))(z - a_k(\varepsilon)), \quad J_\varepsilon = \bigcup_{k=1}^{N+1} (b_{k-1}(\varepsilon), a_k(\varepsilon)).$$

We have normalized in this manner as, when $\varepsilon \rightarrow 0$, the endpoints $(b_k(\varepsilon), a_k(\varepsilon))$ converge to their positions in the full line equilibrium density ψ_V , and $R_\varepsilon(z)$ and J_ε converge to $R(z)$ and J defined in (2.7) and directly above.

The integral in (2.22) over J_ε can be replaced by

$$G_\varepsilon^0(z) = \frac{1}{2} \int_{\mathcal{C}} \frac{iV'(s)/\pi}{\sqrt{R_\varepsilon(s)}} (a_{N+1}(\varepsilon) - s) \frac{ds}{s - z}$$

for \mathcal{C} a clockwise oriented contour surrounding both J_ε and z . Now,

$$\begin{aligned} \int_{\mathcal{C}} \frac{iV'(s)/2\pi}{\sqrt{R_\varepsilon(s)}} (a_{N+1}(\varepsilon) - s) \frac{ds}{s - z} &= (a_{N+1}(\varepsilon) - z) \int_{\mathcal{C}} \frac{iV'(s)/2\pi}{\sqrt{R_\varepsilon(s)}} \frac{ds}{s - z} - \int_{\mathcal{C}} \frac{iV'(s)/2\pi}{\sqrt{R_\varepsilon(s)}} ds \\ &\equiv (a_{N+1}(\varepsilon) - z) h_{V,\varepsilon}(z) + C_\varepsilon, \end{aligned} \quad (2.23)$$

where $h_{V,\varepsilon}(z)$ and C_ε are the same as $h_{V,n,\alpha}(z)$ and $C_{n,\alpha}$ in the statement of the Lemma. Since $h_V(z)$ figuring in the definition of ψ_V is exactly $\lim_{\varepsilon \rightarrow 0} h_{V,\varepsilon}$ (recall (2.6)), it is left to prove that

$$C_\varepsilon = \int_{\mathcal{C}} \frac{V'(s)}{\sqrt{R(s)}} \frac{ds}{2\pi i} - \frac{\varepsilon}{2} \int_{\mathcal{C}} \frac{V'(s)}{\sqrt{R(s)}} \frac{1}{s - 1} \frac{ds}{2\pi i} + \mathcal{O}(\varepsilon^2). \quad (2.24)$$

Indeed,

$$C_0 = \int_{\mathcal{C}} \frac{V'(s)}{\sqrt{R(s)}} \frac{ds}{2\pi i} \equiv 0, \quad (2.25)$$

by a moment condition for the full-line (or “free”) problem, and the second integral in (2.24) is exactly $h_V(1)$. Further, the endpoints $\{b_k(\varepsilon)\}$ and $\{a_k(\varepsilon)\}$ turn out to be real analytic functions of ε , and we have

$$C_\varepsilon = \frac{\varepsilon}{2} \int_{\mathcal{C}} \frac{V'(s)}{\sqrt{R(s)}} \left(\sum_{k=1}^{N+1} \frac{a'_k(0)}{s - a_k(0)} + \frac{b'_{k-1}(0)}{s - b_{k-1}(0)} \right) \frac{ds}{2\pi i} + \mathcal{O}(\varepsilon^2).$$

The advertised $\mathcal{O}(\varepsilon)$ term will then arise from $a_{N+1}(0) = 1$ and $a'_{N+1}(0) = -1$, and additional fact that all other endpoints have vanishing first derivative at $\varepsilon = 0$.

The idea behind verifying these last claims is to view our constrained problem as a perturbation of the free problem at the modulation point P , where $a_{N+1} = 1$. Returning to (2.22) set $\tilde{R}(z) =$

$R(z)/(z - a_{N+1})^2$, suppressing the dependence on $\varepsilon \geq 0$. For the constrained problem, the system of $2N + 1$ modulation equations determining the endpoints are

$$T_j \equiv \int_J \frac{V'(s)}{\sqrt{\tilde{R}(s)}_+} s^j ds = 0, \quad 0 \leq j \leq N - 1, \quad (2.26)$$

$$T_N \equiv \int_J \frac{V'(s)}{\sqrt{\tilde{R}(s)}_+} s^N ds = \frac{1}{\pi i}, \quad (2.27)$$

the so-called moment conditions along with the integral conditions

$$N_k \equiv \int_{\gamma_k} G(z) dz = 0, \quad 1 \leq k \leq N, \quad (2.28)$$

where γ_k denotes the loop around the cut (b_{k-1}, a_k) . Adding the additional relation

$$T_{-1} \equiv \int_J \frac{V'(s)}{\sqrt{\tilde{R}(s)}_+} \frac{ds}{s - a_{N+1}} = 0 \quad (2.29)$$

gives the system for the free problem. In particular, the moment conditions for the full problem read $\int_J V'(s)/\sqrt{R(s)}_+ s^j ds = 0$ for $0 \leq j \leq N$ and $(\pi i)^{-1}$ for $j = N + 1$, and T_{-1} takes you from the system (2.26)–(2.27) to this one.

Our regularity assumption for the free problem implies that the absolute value of the $(2N + 2) \times (2N + 2)$ Jacobian of the map $\{a_j, b_k\} \mapsto \{T_j, N_k\}$ is bounded below by a positive constant at point P . A full proof may be found in Section 12 of [15]. In order to conclude all other endpoints are real-analytic functions of a_{N+1} , and so ε , we must show that the $(2N + 1) \times (2N + 1)$ Jacobian of the modulation equations for the constrained problem is also bounded below by a positive constant at point P . (And then invoke the implicit function theorem). Clearly, this will hold if

$$\frac{\partial T_j}{\partial a_{N+1}} = 0 \text{ for } 0 \leq j \leq N, \text{ and } \frac{\partial N_k}{\partial a_{N+1}} = 0 \text{ for } 1 \leq k \leq N,$$

as, if so, the $(2N + 1) \times (2N + 1)$ minor must be non-degenerate, less the full Jacobian is. Note also that by a simple application of the chain rule, this will imply the vanishing of the first derivatives of all endpoints other than a_{N+1} at $\varepsilon = 0$, a fact we used above.

Finally, $\frac{\partial}{\partial a_{N+1}} T_j = \int_J V'(s)/\sqrt{R(s)}_+ s^j ds$ and the latter vanishes for $j \leq N$ by the moment conditions for the free problem. Also, by linearity the vanishing of the derivative of any N_k will follow from $\frac{\partial G}{\partial a_{N+1}}(P) = 0$. We compute

$$\begin{aligned} \frac{\partial G}{\partial a_{N+1}} &= \frac{-1}{4\pi i} \sqrt{\tilde{R}(z)} \int_J \frac{V'(s)}{\sqrt{R(s)}_+} \frac{ds}{s - z} + \frac{1}{4\pi i} \frac{\sqrt{\tilde{R}(z)}}{(z - a_{N+1})} \int_J \frac{V'(s)(s - a_{N+1})}{\sqrt{R(s)}_+} \frac{ds}{s - z} \\ &= \frac{1}{4\pi i} \frac{\sqrt{\tilde{R}(z)}}{(z - a_{N+1})} \int_J \frac{V'(s)}{\sqrt{R(s)}_+} ds, \end{aligned}$$

and note the last integral vanishes at point P as the integral appearing reduces to that in condition (2.29). \square

3 Steepest descent

3.1 First transformation $Y \mapsto T$

As in [11] Section 3.3, we define $T(z)$ for either $\alpha > 0$ or $\alpha < 0$ by conjugation,

$$T(z) = e^{-\frac{n\ell}{2}\sigma_3} Y(z) e^{\frac{n\ell}{2}\sigma_3} e^{-ng(z)\sigma_3}, \quad (3.1)$$

where $g(z)$ is the log-transform of either ψ_V^\rightarrow or ψ_V^\leftarrow , and we recall that $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The jump matrix for $Y(z)$ is transformed into,

$$V_T(z) = \begin{pmatrix} e^{-n(g_+(z)-g_-(z))} & e^{n(g_+(z)+g_-(z)-V(z)-\ell)} \\ 0 & e^{-n(g_+(z)-g_-(z))} \end{pmatrix}, \quad z \in \Gamma_{n,\alpha}.$$

Next, using the relations (2.9) and (2.12) satisfied by $g(z)$ and the fact that $e^{ng(z)} \approx z^n$ for $z \rightarrow \infty$, we find that $T(z)$ is the unique solution of the following *RHP*.

RHP for T : We seek $T(z)$ analytic in $\mathbb{C} \setminus \Gamma_{n,\alpha}$, with jump relations,

$$\begin{aligned} T_+(z) &= T_-(z) \begin{pmatrix} e^{-n(g_+(z)-g_-(z))} & 1 \\ 0 & e^{n(g_+(z)-g_-(z))} \end{pmatrix}, \quad z \in \bar{J} \\ T_+(z) &= T_-(z) \begin{pmatrix} e^{-2\pi i n \Omega_j} & e^{-n(g_+(z)+g_-(z)-V(z)-\ell)} \\ 0 & e^{2\pi i n \Omega_j} \end{pmatrix}, \quad z \in (a_j, b_j), j = 1, \dots, N, \\ T_+(z) &= T_-(z) \begin{pmatrix} 1 & e^{-n(g_+(z)+g_-(z)-V(z)-\ell)} \\ 0 & 1 \end{pmatrix}, \quad z < b_0 \text{ or } a_{N+1} < z < c_{n,\alpha}, \end{aligned} \quad (3.2)$$

and asymptotics,

$$T(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \quad z \rightarrow \infty. \quad (3.3)$$

That is, we now have an *RHP* normalized at ∞ .

3.2 Second transformation $T \mapsto S$

This step is the descent, transforming the oscillatory diagonal entries of the jump matrices in the *RHP* for $T(z)$ into exponentially decaying off-diagonal entries in an equivalent problem for a function $S(z)$.

For the $\alpha > 0$ case we follow [11] without change. For $z \in \mathbb{C} \setminus \Gamma_{n,\alpha}$ in the region of analyticity of $h_V(z)$, recall (2.6), define

$$\phi(z) = \int_{a_{N+1}}^z R^{1/2}(s) h_V(s) ds,$$

where $a_{N+1} = 1$ and the path of integration does not cross $\Gamma_{n,\alpha}$. From (2.9) and (2.12) we have that, for each $z \in (b_{j-1}, a_j) \subset J$,

$$\begin{aligned} g_+(z) - g_-(z) &= 2\pi i \int_z^{a_{N+1}} \psi_V(s) ds \\ &= \int_z^{a_j} R_+^{1/2}(s) h(s) ds + 2\pi i \int_{b_j}^{a_{N+1}} \psi(s) ds \\ &= -\phi_+(z) = \phi_-(z). \end{aligned} \quad (3.4)$$

That is, $-\phi(z)$ and $\phi(z)$ are analytic continuations of $g_+(z) - g_-(z)$ above and below each band. Also, ϕ_+ and ϕ_- are purely imaginary on each band and an easy exercise using the Cauchy-Riemann conditions shows that,

$$\operatorname{Re} \phi(z) < 0, \text{ for small } \operatorname{Im}(z) \neq 0 \text{ and } \operatorname{Re}(z) \in J. \quad (3.5)$$

Since the g -function tied to ψ_V^- (for $\alpha < 0$) satisfies the same basic relations we can define extensions for $g_+(z) - g_-(z) = 2\pi i \int_z^{c_{n,\alpha}} \psi_V^-(s) ds$, and so ϕ , in the same way.

With these properties of ϕ in mind, the jump contour is deformed off the line by opening a lens around each band based on the factorization

$$\begin{aligned} \begin{pmatrix} e^{-n(g_+(z)-g_-(z))} & 1 \\ 0 & e^{n(g_+(z)-g_-(z))} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ e^{n\phi_-(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{n\phi_+(z)} & 1 \end{pmatrix} \\ &\equiv B_-^{-1}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B_+(z). \end{aligned}$$

Set,

$$S(z) = \begin{cases} T(z), & z \text{ in the exterior of each lens,} \\ T(z)B_{\pm}^{-1}(z), & z \text{ in the upper/lower part of each lens,} \end{cases} \quad (3.6)$$

in which what is meant by a lens, and the resulting contour with lenses $\Sigma_{n,\alpha}$, is spelled out in Figure 2. We are led to:

RHP for S : $S(z)$ is analytic in $\mathbb{C} \setminus \Sigma_{n,\alpha}$, satisfies $S(z) = I + \mathcal{O}(\frac{1}{z})$ as $z \rightarrow \infty$ with $z \notin \Sigma_{n,\alpha}$, along with the jump relations,

$$\begin{aligned} S_+(z) &= S_-(z) \begin{pmatrix} 1 & 0 \\ e^{n\phi(z)} & 1 \end{pmatrix}, & z \in \Sigma_{n,\alpha} \cap \mathbb{C}_{\pm}, \\ S_+(z) &= S_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in J, \\ S_+(z) &= S_-(z) \begin{pmatrix} e^{-2\pi i \Omega_j} & e^{-n(g_+(z)+g_-(z)-V(z)-\ell)} \\ 0 & e^{2\pi i \Omega_j} \end{pmatrix}, & z \in (a_j, b_j), j = 1, \dots, N, \\ S_+(z) &= S_-(z) \begin{pmatrix} 1 & e^{-n(g_+(z)+g_-(z)-V(z)-\ell)} \\ 0 & 1 \end{pmatrix}, & z < b_0 \text{ or } a_{N+1} < z < c_{n,\alpha}. \end{aligned} \quad (3.7)$$

Note that (3.5) implies the factor $e^{n\phi(z)}$ in (3.7) decays exponentially as $n \rightarrow \infty$. Further, in the regular case V the exponent $g_+(z) + g_-(z) - V(z) - \ell$ appearing in the third and fourth jump matrix is strictly negative in the gaps or past the ends of support. It follows that the corresponding entries also decay exponentially as $n \rightarrow \infty$.

3.3 Model Problem

From the discussion at the end of the previous section we expect the leading order asymptotics to be governed by the 2×2 matrix $P^\infty(z)$ which solves the following model problem.

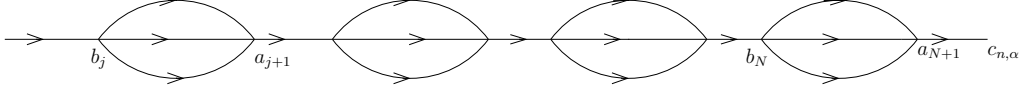


Figure 2: The new contour $\Sigma_{n,\alpha}$ (pictured for $\alpha > 0$) with a lens opened around each band.

RHP for P^∞ : $P^\infty(z)$ is analytic in $\mathbb{C} \setminus [b_0, a_{N+1}]$, $P^\infty(z) = I + \mathcal{O}(1/z)$, as $z \rightarrow \infty$, and

$$\begin{aligned} P_+^\infty(z) &= P_-^\infty(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in J, \\ P_+^\infty(z) &= P_-^\infty(z) \begin{pmatrix} e^{2\pi i \Omega_j} & 0 \\ 0 & e^{-2\pi i \Omega_j} \end{pmatrix}, & z \in (a_j, b_j), \quad j = 1, \dots, N. \end{aligned} \quad (3.8)$$

Though we are led to this problem by considering the $n \rightarrow \infty$ for the jumps of $S(z)$, the bands and gaps over which the jumps of P^∞ are defined should still be taken in their finite n positions for the $\alpha < 0$ case.

While the particulars of P^∞ will not affect the parametrix we eventually construct about $z = 1$, it is required to demonstrate that the above problem does indeed have a solution. Fortunately, this has already been accomplished in [11], where it is proved that (3.8) has a unique solution satisfying $\det P^\infty(z) \equiv 1$.

3.4 Last transformation $S \mapsto R$

Since the convergence of the jumps for $S(z)$ to those for P^∞ is not uniform near the endpoints, we have to perform a local analysis at each of the endpoints a_j, b_j . The analysis at a_{N+1} is particular to the present endeavor and is the subject of the next section, for the rest though we may again refer to [11].

For n large enough, each $x_0 = a_j, b_j$ is regular and there will be no interior “singular” points. Surround each x_0 by a small disk and consider the set of local RHP’s:

$$\begin{aligned} P_{x_0}(z) &\text{ analytic in } \{|z - x_0| < \varepsilon'\} \setminus \Sigma_{n,\alpha} \text{ for a } \varepsilon' > \varepsilon, \\ P_{x_0}(z) \text{ and } S(z) &\text{ share jump conditions on } \Sigma_{n,\alpha} \cap \{|z - x_0| < \varepsilon\}, \\ P_{x_0}(z)(P^\infty)^{-1}(z) &= I + \mathcal{O}(n^{-\kappa}), \text{ uniformly for } |z - x_0| = \varepsilon \text{ with a } \kappa > 0. \end{aligned} \quad (3.9)$$

The last condition matches asymptotics of $P_{x_0}(z)$ to those outside its disk. Granted a solution we define

$$R(z) = \begin{cases} S(z)(P^\infty)^{-1}(z), & z \text{ outside the disks,} \\ S(z)(P_\bullet)^{-1}(z) & z \text{ inside the disks,} \end{cases} \quad (3.10)$$

in which $P_\bullet(z)$ stands in for whichever $P_{x_0}(z)$ corresponds to the given disk. At all $x_0 \neq a_{N+1}$, it is well known $P_\bullet(z)$ is given explicitly in terms of Airy functions; the connected RHP is in fact RHP^\rightarrow with $\alpha = \infty$, and the error exponent is $\kappa = 1$. Assuming that parametrices for $z = a_{N+1}$ exist, $R(z)$ will be analytic off the system of contours described by Figure 3. (While there is an isolated singularity in the second column of $R(z)$ at $a_{N+1} = c_{n,\alpha}$ traced back to that in $Y(z)$ at

the same point, it is logarithmic and so removable). Next, since $S(z)$ and $P^\infty(z)$ are normalized at infinity and $\det P^\infty(z) = 1$, it follows from its definition that $R(z)$ is also normalized at infinity. These facts are key ingredients of the result that, uniformly for $z \in \mathbb{C} \setminus \Sigma_R$,

$$R(z) = I + \mathcal{O}(n^{-\kappa}), \quad n \rightarrow \infty, \quad (3.11)$$

with proof identical to that in [11]. It follows that $R(z)$, and also $\frac{d}{dz}R(z)$, are uniformly bounded for large n , and from (3.10) that $\det(R(z)) \equiv 1$.



Figure 3: Part of the contour Σ_R .

4 Paramatrices at the right edge

At last we build paramatrices for the RHP's ($\alpha > 0$ or $\alpha < 0$) in a neighborhood of $z = a_{N+1}$ as $n \rightarrow \infty$; these are described in terms of the solutions of RHP^\rightarrow and RHP^\leftarrow .

When $\alpha > 0$, we have that $a_{N+1} \equiv 1 < c_{n,\alpha}$. It is convenient at this point to bring in the fact that,

$$g_+(z) + g_-(z) + V(z) - \ell = - \int_1^z R^{1/2}(s) h_V(s) ds = -\phi(z), \quad (4.1)$$

for real z with $|z| > 1$. Then, for a fixed $U_\varepsilon = \{z : |z - 1| < \varepsilon\}$ we require a 2×2 $P^\rightarrow(z)$ which is analytic in $U_\varepsilon \setminus \Sigma_{n,\alpha}$ and satisfies,

$$\begin{aligned} (P^\rightarrow(z))_+ &= (P^\rightarrow(z))_- \begin{pmatrix} 1 & 0 \\ e^{n\phi(z)} & 1 \end{pmatrix}, & z \in (\Sigma_{n,\alpha} \cap U_\varepsilon) \cap \mathbb{C}_\pm, \\ (P^\rightarrow(z))_+ &= (P^\rightarrow(z))_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (1 - \varepsilon, 1), \\ (P^\rightarrow(z))_+ &= (P^\rightarrow(z))_- \begin{pmatrix} 1 & e^{-n\phi(z)} \\ 0 & 1 \end{pmatrix}, & z \in [1, 1 + \frac{\alpha}{c_V n^{2/3}}), \end{aligned} \quad (4.2)$$

with

$$P^\rightarrow(z)(P^\infty(z))^{-1} = I + \mathcal{O}\left(\frac{1}{n}\right), \quad z \in \partial U_\varepsilon \setminus \Sigma_{n,\alpha}. \quad (4.3)$$

The asymptotics (4.3) entail a matching condition between the inner and outer solutions.

If instead $\alpha < 0$, the endpoint depends on n as in $a_{N+1} = 1 + \alpha/c_V n^{2/3} = c_{n,\alpha}$. It is convenient though to keep the same neighborhood U_ε fixed about $z = 1$ and the problem is to find $P^\leftarrow(z)$,

analytic in $U_\varepsilon \setminus \Sigma_{n,\alpha}$ with jump conditions

$$\begin{aligned} (P^\leftarrow(z))_+ &= (P^\leftarrow(z))_- \begin{pmatrix} 1 & 0 \\ e^{n\phi(z)} & 1 \end{pmatrix}, \quad z \in (\Sigma_{n,\alpha} \cap U_\varepsilon) \cap \mathbb{C}_\pm, \\ (P^\leftarrow(z))_+ &= (P^\leftarrow(z))_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (1 - \varepsilon, 1 + \frac{\alpha}{c_V n^{2/3}}), \end{aligned} \quad (4.4)$$

and again,

$$P^\leftarrow(z)(P^\infty(z))^{-1} = I + \mathcal{O}\left(\frac{1}{n}\right), \quad z \in \partial U_\varepsilon \setminus \Sigma_{n,\alpha}. \quad (4.5)$$

These problems are now mapped onto those for $M^\rightarrow(\zeta)$ and $M^\leftarrow(\zeta)$. For $z \in U_\varepsilon$ define two changes of variables $z \rightarrow \zeta = \zeta_n^\rightarrow(z)$, via

$$n\phi(z) = \frac{4}{3}\zeta^{3/2}, \quad \text{for } \alpha > 0, \quad (4.6)$$

and

$$n\phi(z) = \frac{4}{3}\zeta^{3/2} + 2\alpha\zeta^{1/2}, \quad \text{for } \alpha < 0. \quad (4.7)$$

Also, set

$$E_n(z) = P^\infty(z) \frac{1}{\sqrt{2i}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} (\zeta_n(z))^{\sigma_3/4}. \quad (4.8)$$

Lemma 4.1. *The RHPs ((4.2), (4.3)) and ((4.4), (4.5)) are solved by*

$$P^\rightarrow(z) = E_n(z) M_n^\rightarrow(\zeta_n^\rightarrow(z)) \quad \text{and} \quad P^\leftarrow(z) = E_n(z) M^\leftarrow(\zeta_n^\leftarrow(z)). \quad (4.9)$$

Here $\zeta_n^\rightarrow(z)$ and $\zeta_n^\leftarrow(z)$ is given by (4.6) or (4.7) respectively. Also, $M_n^\rightarrow(\zeta)$ represents the solution of RHP $^\rightarrow$ in which the jump over $[0, \alpha)$ is replaced by the same jump over $[0, \alpha_n)$ with $\alpha_n \rightarrow \alpha$ as $n \rightarrow \infty$ defined below in (4.12).

Proof. For $\alpha > 0$, from (2.14) we have that: with z close to 1,

$$n\phi(z) = n \int_1^z (s-1)^{1/2} \beta_V(s) ds = \frac{4}{3} n (s-1)^{3/2} \tilde{\beta}(z), \quad (4.10)$$

where $\tilde{\beta}(z)$ inherits the analyticity properties of β_V . Now set

$$\zeta_n(z) = n^{2/3} (z-1) (\tilde{\beta}(z))^{2/3}. \quad (4.11)$$

Along with being analytic in U_ε , $\tilde{\beta}(1) > 0$, and the branch may be chosen so that, $(\tilde{\beta}(1))^{2/3} > 0$. Then, by choice of ε , $\zeta_n(z)$ maps U_ε one-to-one and onto an open neighborhood of $\zeta = \zeta_n(1) = 0$. Further, $\zeta_n(U_\varepsilon \cap \mathbb{R}) \subset \mathbb{R}$, $\zeta_n(U_\varepsilon \cap \mathbb{C}_\pm) \subset \mathbb{C}_\pm$, and those parts of the z -contour $(\Sigma_{n,\alpha} \cap U_\varepsilon) \cap \mathbb{C}_\pm$ can be chosen so that their images are $\arg \zeta \equiv \pm \frac{2}{3}\pi$. Last, $z \in [1, 1 + \alpha/c_V n^{2/3}]$ is mapped to

$$\zeta \in \left[0, \alpha (\tilde{\beta}(c_{n,\alpha}))^{2/3}/c_V\right] \equiv [0, \alpha_n] \rightarrow [0, \alpha(\beta_V(1)/2)^{2/3}/c_V] = [0, \alpha], \quad (4.12)$$

as $n \rightarrow \infty$, providing the definition of α_n .

When $\alpha < 0$, we have from (2.17) that,

$$\begin{aligned} n\phi(z) &= n \int_{c_{n,\alpha}}^z \left((s - c_{n,\alpha})^{1/2} \beta_{V,n,\alpha}^{(a)}(s) - \alpha c_V^{-1} n^{-2/3} (s - c_{n,\alpha})^{-1/2} \beta_{V,n,\alpha}^{(b)}(s) \right) ds \\ &= \left\{ \frac{4}{3} n (z - c_{n,\alpha})^{3/2} + \left(\frac{2\alpha}{c_V} n^{1/3} + \mathcal{O}(n^{-1/3}) \right) (z - c_{n,\alpha})^{1/2} \right\} \widehat{\beta}(z). \end{aligned} \quad (4.13)$$

Note the change of sign: $(s - c_{n,\alpha})_+^{1/2} = -(s - c_{n,\alpha})_-^{1/2}$. Here again $\widehat{\beta}(z)$ is analytic and positive in a fixed neighborhood of 1, and so $c_{n,\alpha}$, for all large enough n . (That $\beta_{V,n,\alpha}^{(a,b)}(z)$ are analytic near 1 and differ by $\mathcal{O}(n^{-2/3})$ at $z = c_{n,\alpha}$ is used.) Choosing the $2/3$ -root of $\widehat{\beta}$ positive in that same neighborhood, it follows that (4.7) has the required properties: U_ε is mapped one-to-one and onto a neighborhood of $\zeta = \zeta_n(c_{n,\alpha}) = 0$, $\zeta_n(\mathbb{R} \cap U_\varepsilon)$ is real, and the segments of $\Sigma_{n,a}$ can be chosen to map onto $\arg \zeta = \pm \frac{2}{3}\pi$.

Plainly, $M_n^\rightarrow(\zeta_n(z))$ and $M^\leftarrow(\zeta_n(z))$ satisfy the jumps specified for $P^\rightarrow(z)$ and $P^\leftarrow(z)$. Next, as fully explained in [12], $E_n(z)$ is analytic in U_ε and so does not affect the jump relations. Briefly, $P^\infty(z)$ has a singularity of the form $(z - a_{N+1})^{-\sigma_3/4}$ at the right edge, and this is compensated by the appearance of $(\zeta_n(z))^{\sigma_3/4}$ in (4.8) and the fact that $\zeta_n(z) \approx C n^{2/3} (z - a_{N+1})$ whether α is positive or negative. The asymptotics (4.3) and (4.5) follow from the fact that there is a constant $c > 0$ with $|\zeta_n(z)| > c n^{2/3}$ uniformly on $|z| = \varepsilon$ and $n \rightarrow \infty$ and the behavior as $\eta \rightarrow \infty$ of $M^\rightarrow(\zeta)$ stated in (1.18). \square

Remark 4.2. Along with $E_n(z)$ being analytic in a neighborhood of $z = 1$, it also follows from the form of $P^\infty(z)$ that both $E_n(z)$ and $\frac{d}{dz} E_n(z)$ are uniformly bounded in that neighborhood. Further, $\det E_n(z) \equiv 1$ since $\det P^\infty(z) \equiv 1$.

5 Proof of Theorem 1.1

The derivation below borrows heavily from [17]. First note the expression for $L_{n,\alpha}$ in terms of the Y matrix (defined in (2.2)):

$$L_{n,\alpha}(z, w) = -\frac{1}{2\pi i} e^{-\frac{1}{2}V(z) - \frac{1}{2}V(w)} \frac{Y_{11}(z)Y_{21}(w) - Y_{21}(z)Y_{11}(w)}{z - w}. \quad (5.1)$$

We are interested in the asymptotics of the above kernel for real z and w to the right of a_{N+1} , the endpoint of support of the density of states. Unravelling the sequence of transformations leading from Y to R , we have by (3.1), (3.6) and (3.10),

$$Y(z) = Y_{n,\alpha}(z) = \sqrt{2\pi} e^{\frac{\pi i}{4}} e^{\frac{n\ell}{2}\sigma_3} R(z) E_n(z) \mathbf{M}_{n,\alpha}(z) e^{\frac{1}{2}n\phi(z)\sigma_3} e^{-\frac{n\ell}{2}\sigma_3} e^{ng(z)\sigma_3} \quad (5.2)$$

in which we have made the definition:

$$\mathbf{M}_{n,\alpha}(z) = \begin{cases} \frac{e^{\pi i/4}}{\sqrt{2\pi}} M_n^\rightarrow(\zeta_n^\rightarrow(z)) e^{-\frac{1}{2}n\phi(z)\sigma_3} & \text{for } \alpha > 0 \\ \frac{e^{\pi i/4}}{\sqrt{2\pi}} M^\leftarrow(\zeta_n^\leftarrow(z)) e^{-\frac{1}{2}n\phi(z)\sigma_3} & \text{for } \alpha < 0 \end{cases}. \quad (5.3)$$

Here $\zeta_n^\rightarrow(z)$ and ζ_n^\leftarrow are given by (4.6) and (4.7), and M_n^\rightarrow is as in Lemma 4.1. It follows that,²

$$\begin{pmatrix} Y_{11}(z) \\ Y_{21}(z) \end{pmatrix} = \sqrt{2\pi} e^{\frac{\pi i}{4}} e^{n(g(z) - \frac{\ell}{2} + \frac{1}{2}\phi(z))} e^{\frac{n\ell}{2}\sigma_3} R(z) E_n(z) \begin{pmatrix} (\mathbf{M}_{n,\alpha})_{11}(\zeta_n(z)) \\ (\mathbf{M}_{n,\alpha})_{21}(\zeta_n(z)) \end{pmatrix}. \quad (5.4)$$

Now, for either the (\rightarrow) or (\leftarrow) case and real $z > a_{N+1} = 1$ or $a_{N+1} = 1 - \alpha/c_V n^{2/3}$, we have that,

$$g(z) - \frac{1}{2}V(z) - \frac{\ell}{2} = \frac{1}{2}(g_+(z) + g_-(z) - V(z) - \ell) = -\frac{1}{2}\phi(z),$$

recall (4.1). Further, set

$$K(z) \equiv R(z)E_n(z),$$

and recall that both $R(z)$ and $E_n(z)$ have determinant = 1, are analytic in a neighborhood of $z = 1$, and, along with their derivatives, are uniformly bounded there. Obviously, $K(z)$ inherits these properties. Next change variables as in

$$z \mapsto x_n = 1 + \frac{x}{c_V n^{2/3}}, \quad w \mapsto y_n = 1 + \frac{y}{c_V n^{2/3}},$$

with $x, y > \alpha$, and summarizing the steps thus far we have:

$$\begin{aligned} \widehat{L}_{n,\alpha}(x_n, y_n) &\equiv \frac{1}{c_V n^{2/3}} L_{n,\alpha}(x_n, y_n) \\ &= -\frac{1}{2\pi i(x-y)} \det \begin{pmatrix} e^{-\frac{n\ell}{2}} e^{-\frac{1}{2}nV(x_n)} Y_{11}(x_n) & e^{-\frac{n\ell}{2}} e^{-\frac{1}{2}nV(y_n)} Y_{11}(y_n) \\ e^{\frac{n\ell}{2}} e^{-\frac{1}{2}nV(x_n)} Y_{21}(x_n) & e^{\frac{n\ell}{2}} e^{-\frac{1}{2}nV(y_n)} Y_{21}(y_n) \end{pmatrix} \\ &= \frac{1}{(x-y)} \det \left[K(x_n) \begin{pmatrix} (\mathbf{M}_{n,\alpha})_{11}(x_n) & 0 \\ (\mathbf{M}_{n,\alpha})_{21}(x_n) & 0 \end{pmatrix} + K(y_n) \begin{pmatrix} 0 & (\mathbf{M}_{n,\alpha})_{11}(y_n) \\ 0 & (\mathbf{M}_{n,\alpha})_{21}(y_n) \end{pmatrix} \right]. \end{aligned} \quad (5.5)$$

Before further manipulations we record the following two facts.

Claim 5.1. As $n \rightarrow \infty$,

$$\zeta_n^\rightarrow(x_n) = x(1 + \mathcal{O}(n^{-2/3})), \quad \text{and} \quad \zeta_n^\leftarrow(x_n) = (x - \alpha)(1 + \mathcal{O}(n^{-2/3})), \quad (5.6)$$

uniformly for x in compact sets of (α, ∞) . Also,

$$n\phi(x_n) = n\phi_{n,\alpha}(x_n) = \begin{cases} \frac{4}{3}x^{3/2}(1 + \mathcal{O}(n^{-2/3})), & \alpha > 0, \\ (\frac{4}{3}(x - \alpha)^{3/2} + 2\alpha(x - \alpha)^{1/2})(1 + \mathcal{O}(n^{-2/3})) & \alpha < 0, \end{cases} \quad (5.7)$$

with the same uniformity in x as $n \rightarrow \infty$.

Claim 5.2. For all real $x > \alpha$,

$$M_n^\rightarrow(x) - M^\rightarrow(x) = \mathcal{O}(n^{-2/3}), \quad (5.8)$$

and

$$\frac{d}{dx}(M_n^\rightarrow(x) - M^\rightarrow(x)) = \mathcal{O}(n^{-2/3}), \quad (5.9)$$

as $n \rightarrow \infty$. The estimates are uniform for x a positive distance from α .

²If z, w are taken to the left of a_{N+1} , there is an additional factor of $\begin{pmatrix} 1 & 0 \\ e^{-n\phi(z)} & 1 \end{pmatrix}$ in (5.2) arising from opening the lenses in the $T \mapsto S$ step. In that case, Y_{11} and Y_{21} are linear combinations of the first and second rows of $\mathbf{M}_{n,\alpha}$ respectively.

Proofs. The estimate (5.7) follows directly from substituting the definition of x_n into (4.10) and (4.13):

$$n\phi(x_n) = \frac{4}{3}x^{3/2} \times \tilde{\beta}(x_n)/(c_V)^{3/2},$$

for $\alpha > 0$, and,

$$n\phi(x_n) = \left\{ \frac{4}{3}(x - \alpha)^{3/2} + (2\alpha + \mathcal{O}(n^{-4/3}))(x - \alpha)^{1/2} \right\} \times \hat{\beta}(x_n)/(c_V)^{3/2},$$

for $\alpha < 0$. Since $\tilde{\beta}(z)$ and $\hat{\beta}(z)$ are analytic and $= c_V^{3/2} > 0$ at $z = 1$, each of the rightmost factor above may be expanded as in $1 + \mathcal{O}(n^{-2/3}) + \dots$. The same considerations lead to (5.6).

As for (5.8) and (5.9), Lemma 7.2 below proves that $M^\rightarrow(z) = M^\rightarrow(z; \alpha)$ is continuous in α for z supported away from Σ^\rightarrow . The estimate (7.4) obtained in its proof provides

$$M^\rightarrow(z; \alpha) - M^\rightarrow(z; \beta) = \mathcal{O}(\alpha - \beta),$$

with the same holding for the derivative of the left hand side. Now, since $M_n^\rightarrow(z) \equiv M^\rightarrow(z; \alpha_n)$ and (4.12) shows that $\alpha_n = \alpha + \mathcal{O}(n^{-2/3})$, the claim is proven. \square

Picking up the calculation, the matrix within the last determinant of (5.5) is now written as

$$\begin{aligned} K(x_n) \left[\begin{pmatrix} (\mathbf{M}_{n,\alpha})_{11}(x_n) & (\mathbf{M}_{n,\alpha})_{11}(y_n) \\ (\mathbf{M}_{n,\alpha})_{21}(x_n) & (\mathbf{M}_{n,\alpha})_{21}(y_n) \end{pmatrix} \right. \\ \left. + K(y_n)^{-1}(K(x_n) - K(y_n)) \begin{pmatrix} (\mathbf{M}_{n,\alpha})_{11}(x_n) & 0 \\ (\mathbf{M}_{n,\alpha})_{21}(x_n) & 0 \end{pmatrix} \right]. \end{aligned} \quad (5.10)$$

Since the analytic matrix function $K(z)$ satisfies $\det K(z) \equiv 1$, we see that the form of desired limit resides in the first term of (5.10). As for the second term, first note that since $K(z)$ and its derivative are uniformly bounded for $|z - 1| < \varepsilon$ with a small enough $\varepsilon > 0$, it follows that $K(z)^{-1}$ is bounded in kind and that $K(x_n) - K(y_n) = \mathcal{O}(|x_n - y_n|) = \mathcal{O}(|x - y|n^{-2/3})$, by the mean-value theorem. Therefore,

$$K(y_n)^{-1}(K(x_n) - K(y_n)) \begin{pmatrix} (\mathbf{M}_{n,\alpha})_{11}(x_n) & 0 \\ (\mathbf{M}_{n,\alpha})_{21}(x_n) & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{O}(|x - y|n^{-2/3}) & 0 \\ \mathcal{O}(|x - y|n^{-2/3}) & 0 \end{pmatrix}.$$

Along with the estimates on the pre-factor, we are in the domain of analyticity of M_n^\rightarrow and M^\leftarrow , which coupled with (5.6) through (5.8), implies that $\mathbf{M}_{n,\alpha}(x_n)$ is uniformly bounded.

The kernel now reads,

$$\begin{aligned} \hat{L}_{n,\alpha}(x_n, y_n) &= \frac{1}{(x - y)} \det \begin{pmatrix} (\mathbf{M}_{n,\alpha})_{11}(x_n) + \mathcal{O}(|x - y|n^{-2/3}) & (\mathbf{M}_{n,\alpha})_{11}(y_n) \\ (\mathbf{M}_{n,\alpha})_{21}(x_n) + \mathcal{O}(|x - y|n^{-2/3}) & (\mathbf{M}_{n,\alpha})_{21}(y_n) \end{pmatrix} \\ &= \frac{1}{(x - y)} \det \begin{pmatrix} (\mathbf{M}_{n,\alpha})_{11}(x_n) & (\mathbf{M}_{n,\alpha})_{11}(y_n) \\ (\mathbf{M}_{n,\alpha})_{21}(x_n) & (\mathbf{M}_{n,\alpha})_{21}(y_n) \end{pmatrix} + \mathcal{O}(n^{-2/3}), \end{aligned} \quad (5.11)$$

for x, y bounded in (α, ∞) and all large n . Instead of expanding the first term of the right hand side entry-wise, the second column of that matrix is subtracted by the first,

$$\widehat{L}_{n,\alpha}(x_n, y_n) = \frac{1}{(x-y)} \det \begin{pmatrix} (\mathbf{M}_{n,\alpha})_{11}(x_n) - (\mathbf{M}_{n,\alpha})_{11}(y_n) & (\mathbf{M}_{n,\alpha})_{11}(y_n) \\ (\mathbf{M}_{n,\alpha})_{21}(x_n) - (\mathbf{M}_{n,\alpha})_{21}(y_n) & (\mathbf{M}_{n,\alpha})_{21}(y_n) \end{pmatrix} + \mathcal{O}(n^{-2/3}), \quad (5.12)$$

which will allow an estimate uniform in x and y even as $|x-y| \downarrow 0$.

Using (5.7) and (5.6) and the analyticity of $M^\leftarrow(z)$ ($\operatorname{Re} z > 0$), one can check that: with $\xi = \frac{e^{-\pi i/4}}{\sqrt{2\pi}}$,

$$\frac{d}{dx} \left((M_{n,\alpha})_{\cdot 1}(x_n) - (M^\leftarrow)_{\cdot 1}(x-\alpha) \xi e^{\frac{1}{2}\theta(x-\alpha)} \right) = \mathcal{O}\left(\frac{x}{n^{2/3}}\right)$$

with $\theta(x-\alpha) = (4/3)(x-\alpha)^{3/2} + 2\alpha(x-\alpha)^{1/2}$ and $\cdot = 1, 2$. For $\alpha > 0$, the same basic reasoning gives:

$$\frac{d}{dx} \left((M_{n,\alpha})_{\cdot 1}(x_n) - (M_n^\rightarrow)_{\cdot 1}(x) \xi e^{\frac{1}{2}\theta(x)} \right) = \mathcal{O}\left(\frac{x}{n^{2/3}}\right)$$

with $\theta(x) = (4/3)x^{3/2}$ and again $\cdot = 1$ or 2 . It follows that

$$\begin{aligned} (M_{n,\alpha})_{\cdot 1}(x_n) - (M_{n,\alpha})_{\cdot 1}(y_n) &= \xi \left((M^\leftarrow)_{\cdot 1}(x-\alpha) e^{\frac{1}{2}\theta(x-\alpha)} - (M^\leftarrow)_{\cdot 1}(y-\alpha) e^{\frac{1}{2}\theta(y-\alpha)} \right) \\ &\quad + \mathcal{O}(|x-y|n^{-2/3}). \end{aligned} \quad (5.13)$$

for $\alpha < 0$, with the analogous statement for $\alpha > 0$. Detailing how (5.13) is employed in (5.12) we recall the definition of $(f_\alpha^\leftarrow, g_\alpha^\leftarrow)$ from (1.21) and write,

$$\begin{aligned} \widehat{L}_{n,\alpha < 0}(x_n, y_n) &= \frac{1}{x-y} \det \begin{pmatrix} (f_\alpha^\leftarrow(x-\alpha) - f_\alpha^\leftarrow(y-\alpha) + \mathcal{O}(|x-y|n^{-2/3})) & f_\alpha^\leftarrow(y-\alpha) + \mathcal{O}(n^{-2/3}) \\ (g_\alpha^\leftarrow(x-\alpha) - g_\alpha^\leftarrow(y-\alpha) + \mathcal{O}(|x-y|n^{-2/3})) & g_\alpha^\leftarrow(y-\alpha) + \mathcal{O}(n^{-2/3}) \end{pmatrix} \\ &= \mathbb{B}_\alpha(x, y) + \frac{1}{(x-y)} \det \begin{pmatrix} f_\alpha^\leftarrow(x-\alpha) - f_\alpha^\leftarrow(y-\alpha) & \mathcal{O}(n^{-2/3}) \\ g_\alpha^\leftarrow(x-\alpha) - g_\alpha^\leftarrow(y-\alpha) & \mathcal{O}(n^{-2/3}) \end{pmatrix} + \mathcal{O}(n^{-2/3}). \end{aligned} \quad (5.14)$$

The first term of the right hand side is the advertised limit kernel. To show that the second term is $\mathcal{O}(n^{-2/3})$ uniformly in x, y in bounded sets of (α, ∞) note that

$$\frac{f_\alpha^\leftarrow(x-\alpha) - f_\alpha^\leftarrow(y-\alpha)}{x-y} = \mathcal{O}(1),$$

for all such x and y since $x \mapsto (M^\leftarrow)_{11}(x-\alpha)e^{-\frac{1}{2}\phi(x-\alpha)}$ is smooth for $x > \alpha$. The same is true if f_α^\leftarrow is replaced by g_α^\leftarrow . This completes the proof for $\alpha < 0$.

In the case $\alpha > 0$ following the steps behind (5.14) produces

$$\widehat{L}_{n,\alpha > 0}(x_n, y_n) = \frac{\xi^2}{(x-y)} e^{-(\frac{2}{3}x^{3/2} + \frac{2}{3}y^{3/2})} \det \begin{pmatrix} (M_n^\rightarrow)_{11}(x) & (M_n^\rightarrow)_{11}(y) \\ (M_n^\rightarrow)_{21}(x) & (M_n^\rightarrow)_{21}(y) \end{pmatrix} + \mathcal{O}(n^{2/3}).$$

Next repeat the procedure: subtracting the second column from the first in the above determinant and now employing the estimates of Claim 5.2 will allow each appearance of $(M_n^\rightarrow)_{\cdot 1}$ to be replaced with the corresponding $(M^\rightarrow)_{\cdot 1}$ with an overall $\mathcal{O}(n^{-2/3})$ error.

6 Existence for the local problems $RHP^{\vec{c}}$

The general theory connects the the construction of a solution to a given RHP to that of a certain singular integral operator. In particular, consider the RHP (Σ, v) :

$$\begin{aligned} m(\zeta) &\text{ analytic in } \mathbb{C} \setminus \Sigma, \\ m_+(\zeta) &= m_-(\zeta)v(\zeta), \quad \zeta \in \Sigma, \\ m(\zeta) &= I + \mathcal{O}\left(\frac{1}{\zeta}\right), \quad \zeta \rightarrow \infty, \zeta \notin \Sigma, \end{aligned} \tag{6.1}$$

in which v is continuous on Σ away from points of self-intersection and $v(\zeta) \rightarrow I$ as $\zeta \rightarrow \infty$ along Σ . Next define the integral operator

$$C_v f(\zeta) = C_- \left(f(v - I) \right)$$

on $L^2(\Sigma, |d\zeta|)$, where C_- , and C_+ are the \pm -limits of the Cauchy operator:

$$(C_{\pm} f)(\zeta) = \lim_{\substack{\zeta' \rightarrow \zeta \\ \zeta' \in \pm \text{ side of } \Sigma}} \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s - \zeta'} ds, \quad \zeta \in \Sigma.$$

With $v \in I + L^2(\Sigma)$, C_v is bounded from $L^2(\Sigma) \rightarrow L^2(\Sigma)$, and an L^2 solution to (6.1) can be construction out of a solution $\mu_v \in I + L^2(\Sigma)$ of

$$(\mathbb{I} - C_v)\mu_v = I \tag{6.2}$$

via

$$m(\zeta) = I + \frac{1}{2\pi i} \int_{\Sigma} \frac{\mu_v(s)(v(s) - I)}{s - \zeta} ds. \tag{6.3}$$

For details behind these facts, [23] is recommended. Existence for the $RHP(\Sigma, v)$ would then follow from showing that $\mathbb{I} - C_v$ is a bijection in $L^2(\Sigma)$.

To apply this strategy to either RHP^{\rightarrow} or RHP^{\leftarrow} requires a preliminary step: it is not the case that $M^{\vec{c}}$, or their corresponding jump matrices $V^{\vec{c}}$, are normalized to the identity at infinity. Therefore, we bring in

$$\mathbf{m}(\zeta) = \zeta^{\frac{-\sigma_3}{4}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{i\pi}{4}\sigma_3}, \tag{6.4}$$

the fundamental solution of the twist problem:

$$\mathbf{m}_+(\zeta) = \mathbf{m}_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } \zeta \in \mathbb{R}_-,$$

and $\mathbf{m}(\zeta)$ analytic in $\mathbb{C} \setminus \mathbb{R}_-$. With this, we define

$$\widetilde{M}^{\vec{c}}(\zeta) = \begin{cases} M^{\vec{c}}(\zeta) \mathbf{m}^{-1}(\zeta), & \text{for } |\zeta| > R, \\ M^{\vec{c}}(\zeta), & \text{for } |\zeta| < R, \end{cases} \tag{6.5}$$

with a fixed large and positive R . The contours for the pair of RHPs for $\widetilde{M}^{\rightarrow}$ appear in Figure 4. Obviously, in each case the jump along the negative real axis has been removed far out. Further, along $\widetilde{\Sigma}_2$ and $\widetilde{\Sigma}_4$ and $|\zeta| > R$, the new jumps

$$\widetilde{V}(\zeta) = \mathbf{m}(\zeta) \begin{pmatrix} 1 & 0 \\ e^{\theta(\zeta)} & 1 \end{pmatrix} (\mathbf{m}(z))^{-1}, \quad (6.6)$$

with $\theta(\zeta) = \frac{4}{3}\zeta^{2/3}$ or $\theta(\zeta) = \frac{4}{3}\zeta^{2/3} - 2|\alpha|\zeta^{1/2}$, still decay exponentially fast to the identity as $\zeta \rightarrow \infty$. Last, along the introduced contour $|\zeta| = R$, both problems have the uniformly bounded jump $\mathbf{m}^{-1}(\zeta)$.

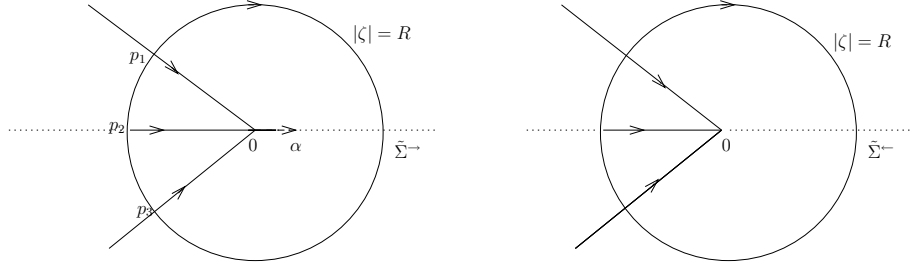


Figure 4: The contours for $(\widetilde{\Sigma}^{\rightarrow}, \widetilde{V}^{\rightarrow})$.

We now have a pair of problems which fit into the above program (jumps are $\in L^\infty \cap (I + L^2)$). The proof of existence now comes in three steps: to show $\mathbb{I} - C_{\widetilde{V}^{\rightarrow}}$ is Fredholm, has zero index, and then that $\ker(\mathbb{I} - C_{\widetilde{V}^{\rightarrow}}) = 0$. This last point is established through a vanishing lemma similar in spirit to [11], Section 5.

6.1 Fredholmness

Fredholmness is implied by the following continuity condition holding throughout the contour. Moving clockwise about a point p on Σ , at which segments of the contour Σ_1 through Σ_k with jumps v_1 through v_k meet, continuity at p is equivalent to

$$I = v_1(p)^{\pm 1} v_2(p)^{\pm 1} \cdots v_k(p)^{\pm 1}, \quad (6.7)$$

in which the sign (± 1) in the exponent is determined by whether the given contour points into, or out of, p . Additionally, this assessment is invariant of conjugations or deformations, see [23].

6.1.1 Criteria (6.7) for RHP^{\rightarrow}

The conjugation by $\mathbf{m}(\zeta)$ in the exterior of a large disk not only produced a problem with decay, from the point of view of Fredholmness, it removed the discontinuity point at $\zeta = \infty$. While new point of intersection $\zeta = p_1, p_2$ and p_3 have been introduced to resulting contour by this move, each corresponded to a point of continuity for the $(\Sigma^{\leftarrow}, V^{\leftarrow})$ and thus remain so by the discussion above.

The worrisome points left on $\widetilde{\Sigma}^\rightarrow$ are then the origin and $\zeta = \alpha$. At the origin, the jump matrices $\widetilde{\Sigma}_1^\rightarrow$ through $\widetilde{\Sigma}_4^\rightarrow$ satisfy,

$$I = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and therefore (6.7) holds. At the point $\zeta = \alpha$, fix a small $\varepsilon > 0$ ($\varepsilon < \alpha$) and consider the local problem

$$(P_\alpha(\zeta))_+ = \begin{cases} (P_\alpha(\zeta))_- \begin{pmatrix} 1 & e^{-\frac{4}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix} & \text{for } \zeta \in [\alpha - \varepsilon, \alpha] \\ (P_\alpha(\zeta))_- & \text{for } \zeta \in (\alpha, \alpha + \varepsilon] \end{cases}.$$

The jump being upper-triangular allows us to write down an explicit solution:

$$P_\alpha(\zeta) = \begin{pmatrix} 1 & \frac{1}{2\pi i} \int_{\alpha-\varepsilon}^{\alpha} \frac{e^{-\frac{4}{3}s^{3/2}}}{s-\zeta} ds \\ 0 & 1 \end{pmatrix}, \quad (6.8)$$

holding in L^2 and in the sense of continuous boundary values away from $\zeta = \alpha - \varepsilon$ and $\zeta = \alpha$. (Recall, $C_+ - C_- = \mathbb{I}$.) Next choose a positive $\varepsilon' < \varepsilon$ and define

$$M_\alpha(\zeta) = \widetilde{M}^\rightarrow(\zeta) P_\alpha(\zeta) \quad \text{for } |\zeta - \alpha| < \varepsilon',$$

leaving $\widetilde{M}^\rightarrow(\zeta)$ unchanged in the exterior of this disk. The effect of conjugating out the local solution is a new *RHP* with contour depicted below in Figure 5. The point of discontinuity $\zeta = \alpha$ has been removed, with the introduced point of self-intersection at $p = p_4$ again automatically a continuity point, having arose from such by way of a conjugation.

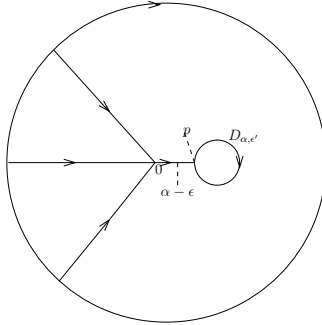


Figure 5: The introduced jump along the boundary of the disk $D_{\alpha, \varepsilon'}$ is P_α^{-1} .

6.1.2 Criteria (6.7) for RHP^\leftarrow

From the previous discussion for RHP^\rightarrow it is plain that we must only deal with the point $\zeta = 0$ where the three ray of the contour Σ^\leftarrow come together. This is again handled by conjugating out a local solution local solution $P_0(\zeta)$ of the *RHP* with jump V^\leftarrow restricted to $\Sigma^\leftarrow \cap (U_\varepsilon = \{\zeta : |\zeta| < \varepsilon\})$.

Given P_0 we set $M_0(\zeta) = \widetilde{M}^\leftarrow(\zeta) P_0(\zeta)$, for $\zeta \in U_{\varepsilon'}$ with $\varepsilon' < \varepsilon$, and $M_0(\zeta) = \widetilde{M}^\leftarrow(\zeta)$ for $\zeta \in \mathbb{C} \setminus U_{\varepsilon'}$. The *RHP* for M_0 will satisfy (6.7), and it will follow that $(\widetilde{\Sigma}^\leftarrow, \widetilde{V}^\leftarrow)$ is Fredholm.

To construct $P_0(\zeta)$, consider

$$\widetilde{M}^\leftarrow(\zeta) e^{-(\frac{2}{3}\zeta^{3/2} + \alpha\zeta^{1/2})\sigma_3} = \widetilde{M}^\leftarrow(\zeta) e^{-\frac{1}{2}\theta(\zeta)\sigma_3}, \quad \zeta \in U_\varepsilon,$$

which has the constant jumps,

$$\begin{aligned} e^{\frac{1}{2}\theta(\zeta)\sigma_3} \begin{pmatrix} 1 & 0 \\ e^{\theta(\zeta)} & 1 \end{pmatrix} e^{-\frac{1}{2}\theta(\zeta)\sigma_3} &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \zeta \in (\Sigma^\leftarrow \cap \mathbb{C}_\pm) \cap U_\varepsilon, \\ e^{\frac{1}{2}\theta_-(\zeta)\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{-\frac{1}{2}\theta_+(\zeta)\sigma_3} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \zeta \in \mathbb{R}_- \cap U_\varepsilon. \end{aligned} \quad (6.9)$$

Extending the jump contours on the right to infinity, we obtain the problem: find some $Q(\zeta)$, which satisfies

$$\begin{aligned} Q_+(\zeta) &= Q_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \zeta \in \mathbb{R}_-, \\ Q_+(\zeta) &= Q_-(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \arg \zeta = \pm \frac{2}{3}\pi, \end{aligned} \quad (6.10)$$

with $Q(\zeta)$ otherwise analytic. From [16], Section 6, we have:

Proposition 6.1. *With $H_0^{(1)}(\cdot)$ and $H_0^{(2)}(\cdot)$ denoting the Hankel functions of the first and second kind, and $I_0(\cdot)$ and $K_0(\cdot)$ the usual modified Bessel functions, the (un-normalized) RHP (6.10) has the solution:*

$$Q(\zeta) = \begin{pmatrix} I_0(\sqrt{\zeta}) & \frac{i}{\pi} K_0(\sqrt{\zeta}) \\ 2\pi i \sqrt{\zeta} I_0'(\sqrt{\zeta}) & -2\sqrt{\zeta} K_0'(\sqrt{\zeta}) \end{pmatrix}. \quad (6.11)$$

for $-\frac{2}{3}\pi < \arg \zeta < \frac{2}{3}\pi$,

$$Q(\zeta) = \begin{pmatrix} \frac{1}{2} H_0^{(1)}(\sqrt{-\zeta}) & \frac{1}{2} H_0^{(2)}(\sqrt{-\zeta}) \\ \pi \sqrt{\zeta} (H_0^{(1)})'(\sqrt{-\zeta}) & \pi \sqrt{\zeta} (H_0^{(2)})'(\sqrt{-\zeta}) \end{pmatrix}, \quad (6.12)$$

for $\frac{2}{3}\pi < \arg \zeta < \pi$, and

$$Q(\zeta) = \begin{pmatrix} \frac{1}{2} H_0^{(1)}(\sqrt{-\zeta}) & -\frac{1}{2} H_0^{(2)}(\sqrt{-\zeta}) \\ -\pi \sqrt{\zeta} (H_0^{(1)})'(\sqrt{-\zeta}) & \pi \sqrt{\zeta} (H_0^{(2)})'(\sqrt{-\zeta}) \end{pmatrix}, \quad (6.13)$$

for $-\pi < \arg \zeta < -\frac{2}{3}\pi$.

It follows that we can set $P_0(\zeta) = Q(\zeta) e^{\frac{1}{2}\theta(\zeta)\sigma_3}$ to form the needed local solution in $|\zeta| < \varepsilon$ and complete the proof.

6.2 Index zero

One consequence of the Gohberg-Krein theory of factorization of matrix-valued functions, is that, given Fredholmness, the index of $\mathbb{I} - C_V$ equals the winding number of $\det V$ over the contour, see [18]. But with V equal to either \widetilde{V}^\leftarrow or $\widetilde{V}^\rightarrow$, $\det V \equiv 1$ and so that winding number is zero.

6.3 Vanishing Lemma

Finally we show that $\ker(\mathbb{I} - C_{\tilde{V}^{\leftrightarrow}}) = 0$, first reverting back to the problem(s) tamed at infinity $(\tilde{\Sigma}^{\leftrightarrow}, \tilde{V}^{\leftrightarrow})$, recall (6.5) and (6.6). Suppose that $\ker(\mathbb{I} - C_{\tilde{V}^{\leftrightarrow}}) \neq 0$. Then there exists $\mu_0^{\leftrightarrow} \in L^2(\tilde{\Sigma})$ which satisfy

$$(\mathbb{I} - C_{\tilde{V}^{\leftrightarrow}})\mu_0^{\leftrightarrow} = 0,$$

and so

$$\widetilde{M}_0^{\leftrightarrow}(\zeta) = \int_{\tilde{\Sigma}^{\leftrightarrow}} \frac{\mu_0^{\leftrightarrow}(s)(I - (\tilde{V}^{\leftrightarrow}(s))^{-1})}{s - \zeta} ds$$

are L^2 -solutions of the *RHPs*:

$$\begin{aligned} \widetilde{M}_0^{\leftrightarrow}(\zeta) & \text{ analytic in } \mathbb{C} \setminus \tilde{\Sigma}^{\leftarrow} \text{ or } \mathbb{C} \setminus \tilde{\Sigma}^{\rightarrow}, \\ (\widetilde{M}_0^{\leftrightarrow})_+(\zeta) &= (\widetilde{M}_0^{\leftrightarrow})_-(\zeta) \tilde{V}^{\leftrightarrow}(\zeta), \quad \zeta \in \tilde{\Sigma}^{\leftarrow} \text{ or } \zeta \in \tilde{\Sigma}^{\rightarrow} \\ \widetilde{M}_0^{\leftrightarrow}(\zeta) &= \mathcal{O}\left(\frac{1}{\zeta}\right), \quad \zeta \rightarrow \infty, \zeta \notin \tilde{\Sigma}^{\leftarrow} \text{ or } \zeta \notin \tilde{\Sigma}^{\rightarrow}. \end{aligned} \quad (6.14)$$

Given this assessment, undoing the transformation that took us from the *RHPs* $(\Sigma^{\leftrightarrow}, V^{\leftrightarrow})$, $(\tilde{\Sigma}^{\leftrightarrow}, \tilde{V}^{\leftrightarrow})$, (removing the conjugation by $\mathbf{m}(\zeta)$ produces M_0^{\rightarrow} and M_0^{\leftarrow} which solve *RHP* $^{\rightarrow}$ or *RHP* $^{\leftarrow}$ with new asymptotics:

$$M_0^{\rightarrow}(\zeta) = \mathcal{O}\left(\frac{1}{\zeta}\right) \zeta^{-\frac{\sigma_3}{4}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{i\pi}{4}\sigma_3}, \quad \zeta \rightarrow \infty, \quad (6.15)$$

holding uniformly in directions respecting $\text{dist}(\zeta, \Sigma^{\leftarrow}) > \delta$ or $\text{dist}(\zeta, \Sigma^{\rightarrow}) > \delta$. We show that the only conclusion is that $M^{\rightarrow}(\zeta) \equiv 0$.

6.3.1 Vanishing lemma for *RHP* $^{\rightarrow}$

The first step is to fold (and twist) the jumps down to the real line, defining a new matrix $Z(\zeta)$ via

$$\begin{aligned} Z(\zeta) &= M_0^{\rightarrow}(\zeta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & 0 < \arg \zeta < \frac{2}{3}\pi \\ Z(\zeta) &= M_0^{\rightarrow}(\zeta) \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \frac{2}{3}\pi < \arg \zeta < \pi, \\ Z(\zeta) &= M_0^{\rightarrow}(\zeta) \begin{pmatrix} 1 & 0 \\ -e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}, & -\pi < \arg \zeta < -\frac{2}{3}\pi, \\ Z(\zeta) &= M_0^{\rightarrow}(\zeta), & -\frac{2}{3}\pi < \arg \zeta < 0. \end{aligned}$$

Then $Z(\zeta)$ is an L^2 -solution of the equivalent *RHP*:

$$\begin{aligned} Z_+(\zeta) &= Z_-(\zeta) \begin{pmatrix} 1 & -e^{\frac{4}{3}\zeta_+^{3/2}} \\ e^{\frac{4}{3}\zeta_-^{3/2}} & 0 \end{pmatrix}, & \zeta \in (-\infty, 0], \\ Z_+(\zeta) &= Z_-(\zeta) \begin{pmatrix} e^{-\frac{4}{3}\zeta^{3/2}} & -1 \\ 1 & 0 \end{pmatrix}, & \zeta \in (0, \alpha], \\ Z_+(\zeta) &= Z_-(\zeta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \zeta \in (\alpha, \infty), \end{aligned} \quad (6.16)$$

with now $Z(\zeta) = \mathcal{O}(\zeta^{-3/2})$, compare (6.15). Denoting the piece-wise defined jump matrix in (6.16) as V , we notice that,

$$0 = \int_{\mathbb{R}} Z_+(s) Z_-^*(s) ds = \int_{\mathbb{R}} Z_-(s) V(s) Z_-^*(s) ds. \quad (6.17)$$

The first equality holds for any functions in the range of C_+ and C_- as may be seen by rational approximation. Adding (6.17) to its conjugate transpose we also find that,

$$0 = \int_{-\infty}^0 Z_-(s) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Z_-^*(s) ds + \int_0^\alpha Z_-(s) \begin{pmatrix} e^{-\frac{4}{3}s^{3/2}} & 0 \\ 0 & 0 \end{pmatrix} Z_-^*(s) ds.$$

It is immediate that the first column of Z_- vanishes *a.e.* on $(-\infty, \alpha]$, and so $(Z_{11}, Z_{21}) = 0$ throughout the lower half plane by analyticity (it lies in the Hardy class H^2). From the structure of the jump across \mathbb{R} one may next conclude that the second column of Z_+ equals 0 *a.e.* on $(-\infty, \alpha)$, and by the same reasoning $(Z_{12}, Z_{22}) = 0$ in the upper half plane.

If we now set,

$$a(\zeta) = Z_{11}(\zeta) \text{ in } \mathbb{C}_+ \quad \text{and} \quad a(\zeta) = Z_{12}(\zeta) \text{ in } \mathbb{C}_-,$$

we are led to the the scalar *RHP*:

$$\begin{aligned} a(\zeta) &\text{ analytic in } \mathbb{C} \setminus \mathbb{R}_-, \\ a_+(\zeta) &= a_-(\zeta) e^{\frac{4}{3}\zeta^{3/2}}, \quad \zeta \in \mathbb{R}_-, \\ a(\zeta) &= \mathcal{O}(\zeta^{-3/4}), \quad \zeta \in \mathbb{C}/\mathbb{R}_-. \end{aligned} \quad (6.18)$$

Note that setting $a = Z_{21}$ in \mathbb{C}_+ and $= Z_{22}$ in \mathbb{C}_- produces the identical *RHP*. If we can conclude that $a(\zeta) \equiv 0$, then $Z(\zeta)$ and so $M_0^+(\zeta)$ also vanish identically, proving that $\ker(\mathbb{I} - C_{V+}) = 0$.

Lemma 6.2. *The unique solution to (6.18) is $a(\zeta) = 0$.*

Proof. Up to this point the jump condition has been understood in the sense of L^2 . To go further it is required that $a_\pm(\zeta)$ are uniformly bounded on the negative real axis. First, since $e^{\frac{4}{3}\zeta^{3/2}}$ is analytic off \mathbb{R}_- , one actually has analytic extensions of a_+ below \mathbb{R}_- and a_- above. That is, (6.18) holds in the sense of continuous boundary values. Further, $e^{\zeta^{3/2}}$ decays as ζ moves below \mathbb{R}_- and $e^{-\zeta^{3/2}} = e^{\zeta_+^{3/2}}$ decays as ζ moves above \mathbb{R}_- , and thus both extensions exhibit at least the same decay as $a(\zeta)$ itself as $\zeta \rightarrow \infty$.

Taking the extension of $a_+(\zeta)$ into a region $\pi < \arg(\zeta) < \pi + \varepsilon$, the Cauchy integral formula provides the representation

$$a_+(\zeta) = \int_{\mathfrak{C}} \frac{\tilde{a}(z)}{z - \zeta} \frac{dw}{2\pi i}, \quad \zeta \in \mathbb{R}_-, \quad (6.19)$$

in which

$$\mathfrak{C} = \{z : \arg(z) = -\pi + \varepsilon/2\} \cup \{z : \arg(z) = \pi - \varepsilon/2\} = \mathfrak{C}^- \cup \mathfrak{C}^+,$$

oriented counter-clockwise, and $\tilde{a}(z) = a(z)$, $a(z)e^{-z^{3/2}}$ on \mathfrak{C}^- , \mathfrak{C}^+ . It follows that for all $\zeta \in (-\infty, \delta]$ $|a_+(\zeta)|$ is bounded by a constant depending only on $\delta > 0$. An identical argument pertains to $a_-(\zeta)$. To achieve a bound down to $\zeta = 0$, we need only note that the jump $= e^{\zeta^{3/2}}$ for $\zeta < 0$

and $= 1$ for $\zeta \geq 0$ is Hölder continuous across zero and the Cauchy transform preserves Hölder continuity.

Granted that $a(\zeta)$ is bounded down to \mathbb{R}_- from both directions, consider now the effect of performing both the above extensions: a_+ below to an angle $\pi + \pi\nu/2$ and a_- above to an angle $-\pi - \pi\nu/2$ with small $\nu > 0$. The resulting function, denoted by $b(\zeta)$, can be made to live on a subset \mathbb{A} of the Riemann surface \mathbb{K} formed by gluing together three copies of \mathbb{C} cut across \mathbb{R}_- in the obvious fashion (alleviating the fact that the initial domain swept out a region of angle $> 2\pi$).

Next bring in the transformation $\zeta(\omega) = \omega^{2+\nu}$ which maps the right half of the ω -plane onto \mathbb{A} , taking the positive/negative imaginary axes in ω onto the lower/upper boundaries of \mathbb{A} . Then $\hat{b}(\omega) = b(\zeta(\omega))$ is analytic in the open half-plane $\{\omega : \Re(\omega) > 0\}$, bounded in the closed half-plane $\{\omega : \Re(\omega) \geq 0\}$, and along the boundary satisfies, $|b(is)| \leq Ce^{-c|s|^{3/2(2+\eta)}} \leq \tilde{C}e^{-\tilde{c}|s|}$. A theorem of Carlson ([19] p. 236) then implies that $\hat{b} \equiv 0$ in the right half plane, which is to say that $a \equiv 0$. \square

6.3.2 Vanishing lemma for RHP^+

The steps for RHP_- are mimicked to the point that the needed conclusion hinges on the following.

Lemma 6.3. *The unique solution of the scalar RHP ,*

$$\begin{aligned} a(\zeta) & \text{ analytic in } \mathbb{C} \setminus \mathbb{R}_-, \\ a_+(\zeta) &= a_-(\zeta) e^{\frac{4}{3}\zeta^{3/2} + 2\alpha\zeta^{1/2}}, \quad \zeta \in \mathbb{R}_-, \\ a(\zeta) &= \mathcal{O}(\zeta^{-3/4}), \quad \zeta \in \mathbb{C}/\mathbb{R}_-, \end{aligned} \tag{6.20}$$

is $a \equiv 0$.

Proof. The analysis of (6.20) is really no different than (6.18). The jump is again analytic in \mathbb{C}/\mathbb{R}_- , implying that a is continuous and uniformly bounded down to \mathbb{R}_- by the same type of extension argument. Since $\alpha > 0$,

$$|e^{\frac{4}{3}\zeta^{3/2} - 2|\alpha|\zeta^{1/2}}| < 1 \quad \text{for } \{\zeta : \arg(\zeta) \in (\pi - \varepsilon, \pi) \cup (-\pi, -\pi + \varepsilon)\}$$

and

$$|e^{\frac{4}{3}\zeta^{3/2} - 2|\alpha|\zeta^{1/2}}| \leq e^{-c s^{3/2}} \quad \text{for } c > 0 \text{ and } \zeta = s e^{i(\pm\pi \mp \varepsilon)}, \quad s \rightarrow \infty,$$

thus extensions share the same qualitative features as above. The analogs of b and \hat{b} are then constructed as before and subject to the same conclusions. \square

7 Properties of the solutions

We prove a continuity result for $M^+(\zeta)$ and $M^-(\zeta)$ in the parameter α and establish asymptotics of those matrix functions for $\alpha \rightarrow \pm \infty$; these will lead to Theorems 1.3 and 1.4.

7.1 Continuity

The continuity result is based on verifying the condition of the following general fact; see Corollary 7.103 of [8] for a proof.

Proposition 7.1. *Consider a family of (uniquely solvable) RHP's on a fixed contour, (Σ, v_n) , $n = 1, 2, \dots$. Assume the existence of a v_∞ , such that the RHP (Σ, v_∞) possesses a unique solution and*

$$\|v_n - v_\infty\|_{L^\infty(\Sigma) \cap L^2(\Sigma)} \rightarrow 0, \quad n \rightarrow \infty. \quad (7.1)$$

Then,

$$\|(m_n)_\pm - (m_\infty)_\pm\|_{L^2(\Sigma)} \rightarrow 0, \quad \text{and} \quad \|(m_n)(z) - (m_\infty)(z)\|_{L^\infty(\mathcal{A})} \rightarrow 0, \quad (7.2)$$

for $n \rightarrow \infty$ and any set \mathcal{A} which is a positive distance from Σ .

The condition (7.1) implies that $\mu_{v_n} = (\mathbb{I} - C_{v_n})^{-1}I$ satisfies $\|\mu_{v_n} - \mu_{v_\infty}\|_{L^2(\Sigma)} \rightarrow 0$. From the expression (6.3), the statements of (7.2) easily follow; the first because C_\pm map L^2 to L^2 . Further, one sees that an estimate of the second type holds for the derivatives, $\frac{d}{dz}[(m_n)(z) - (m_\infty)(z)]$. This is the fact referred to in the proof of Theorem 1.1.

Lemma 7.2. *The condition (7.1) is satisfied by the RHP's $(V^\rightarrow, \Sigma^\rightarrow)$, the parameter α playing the role of n in the Proposition. Continuity holds in each problem down to $\alpha = 0$.*

Remark 7.3. *Note that the problem RHP^\rightarrow has α -dependence in the contour itself via the segment $[0, \alpha)$. In this case, for the L^2 continuity of the boundary values $(M^\rightarrow)_\pm = (M_\alpha^\rightarrow)_\pm$, we show continuity in $L^2(\widehat{\Sigma}^\rightarrow)$ where in $\widehat{\Sigma}^\rightarrow$, the segment $[0, \alpha)$ is extended to $[0, \alpha')$ for any $\alpha' > \alpha$.*

Proof of Lemma 7.2 for RHP^\rightarrow . To employ the conditions of Proposition 7.1 a preliminary conjugation is made to move the dependence of the problem on $\alpha \geq 0$ from the contour into the jump.

Consider first the continuity at a point $\alpha > 0$. In this case the conjugation is affected by the same parametrices used in the poof of existence. Set aside a neighborhood of α , $U_{\alpha, \varepsilon} = (\alpha - \varepsilon, \alpha + \varepsilon)$ for $\varepsilon > 0$ with $\varepsilon \ll \alpha$. Fix also positive a and b with $a < b < \alpha - \varepsilon$ and disks D_a, D_b enclosing $\alpha + \varepsilon$ ($D_b \subset D_a$). Within D_a , and for any $\beta \in U_{\alpha, \varepsilon}$, we have the parametrices,

$$P_\beta(\zeta) = \begin{pmatrix} 1 & \frac{1}{2\pi i} \int_a^\beta \frac{e^{-\frac{4}{3}s^{3/2}}}{s-\zeta} ds \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & C_\beta(\zeta) \\ 0 & 1 \end{pmatrix}.$$

That is, $P_\beta(\zeta)$ satisfies the jump condition across $a < \zeta < \beta$. Next define,

$$\widetilde{M}_\beta(\zeta) = \begin{cases} M^\rightarrow(\zeta) P_\beta(\zeta), & \zeta \in D_b, \\ M^\rightarrow(\zeta), & \zeta \in \mathbb{C} \setminus D_b. \end{cases} \quad (7.3)$$

Now for all β in the defined range we have a family of RHP's on the same contour, with the dependence of β occurring only in the jump

$$\widetilde{V}_\beta(\zeta) = P_\beta^{-1}(\zeta), \quad \text{for } \zeta \in \partial D_b.$$

Also, for all $\zeta \in \partial D_b$ except $\zeta = b$,

$$|C_\beta(\zeta) - C_\alpha(\zeta)| \leq \left| \int_\beta^\alpha \frac{e^{-4/3s^{3/2}}}{s - \zeta} ds \right| = \mathcal{O}(\beta - \alpha), \quad (7.4)$$

there being a positive distance ζ separating and the interval between α and β in this case. On the other hand $(C_\beta)_\pm(b) = (C_\alpha)_\pm(b)$. Thus, the above L^∞ estimate leads to an $L^2(\partial D_b)$ of the same order.

It follows see that \widetilde{M}_β satisfies the criteria of Proposition 7.1, and so $M^\rightarrow(\zeta)$ is continuous at any $\alpha > 0$ in the sense of its boundary values in $L^2(\Sigma^\rightarrow \cap (\mathbb{C} \setminus D_b))$, and also in L^∞ for ζ exterior to D_b and away from Σ^\rightarrow . This already gives the type of continuity claimed in Theorem 1.3 and Claim 5.2 used in the proof of Theorem 1.1.

We complete the analysis by showing the boundary data is L^2 -continuous in the interior D_b . Inverting the move (7.3) we have.

$$\begin{aligned} M^\rightarrow(\zeta) &= \widetilde{M}_\beta(\zeta) P_\beta^{-1}(\zeta) \\ &= -\widetilde{M}_\beta(\zeta) C_\beta(\zeta) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \widetilde{M}_\beta(\zeta). \end{aligned} \quad (7.5)$$

Consider the \pm -limits of the right hand: we want to show they are continuous in $L^2[b, \alpha + \varepsilon]$ as β ranges in $U_{\alpha, \varepsilon}$. First, $\widetilde{M}_\beta(\zeta)$ is analytic inside of D_b with continuous boundary values along ∂D_b , excepting the point $\zeta = b$. It therefore lies in L^∞ of that interval, and the conclusions above include that $\beta \rightarrow \widetilde{M}_\beta(\zeta)$ is continuous in $L^2[b, \alpha + \varepsilon]$. A look at the second line of (7.5) explains that it remains to show that $(C_\beta)_\pm$ are continuous in $L^2[b, \alpha + \varepsilon]$. But, taking $\beta \downarrow \alpha$ from above without any loss of generality,

$$\|(C_\beta)_\pm - (C_\alpha)_\pm\|_{L^2[b, \alpha + \varepsilon]}^2 = \int_\alpha^\beta e^{-\frac{8}{3}s^{3/2}} ds + \frac{1}{4\pi^2} \int_\beta^{\alpha + \varepsilon} \left| \int_\alpha^\beta \frac{e^{-\frac{4}{3}s^{3/2}}}{s - t} ds \right|^2 dt \rightarrow 0.$$

Here, the Hölder continuity of $e^{-4/3s^{3/2}}$ produces the vanishing of the integral over $[b, \alpha]$, and the first term on the right follows from $\|C_\pm \diamond \|_{L^2} \leq \| \diamond \|_{L^2}$. This completes the proof for $\alpha > 0$.

Turning to the case $\alpha = 0$, the first step is to delay the jumps to the left of the origin in the original problem $(V^\rightarrow, \Sigma^\rightarrow)$ by considering the equivalent *RHP*:

$$\begin{aligned} \widehat{M}_+(\zeta) &= \widehat{M}_-(\zeta) \begin{pmatrix} 1 & e^{-\frac{4}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix}, & \zeta \in [0, \beta], \\ \widehat{M}_+(\zeta) &= \widehat{M}_-(\zeta) \begin{pmatrix} e^{\frac{4}{3}\zeta_+^{3/2}} & 1 \\ 0 & e^{-\frac{4}{3}\zeta_-^{3/2}} \end{pmatrix}, & \zeta \in [-1, 0], \\ \widehat{M}_+(\zeta) &= \widehat{M}_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \zeta \in (-\infty, -1), \\ \widehat{M}_+(\zeta) &= \widehat{M}_-(\zeta) \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}, & \zeta \in \{\zeta : \arg(1 + \zeta) = \pm \frac{2}{3}\pi\}, \end{aligned}$$

where $\widehat{M}(\zeta)$ is otherwise analytic and equals $(I + \mathcal{O}(\zeta^{-1}))\mathbf{m}(\zeta)$ as $\zeta \rightarrow \infty$ (recall (6.4)). This problem is obtained from $(V^\rightarrow, \Sigma^\rightarrow)$ by setting

$$\widehat{M}(\zeta) = M^\rightarrow(\zeta) \begin{pmatrix} 1 & 0 \\ \pm e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}, \quad \zeta \in \mathcal{W}_\pm,$$

where \mathcal{W}_\pm is the intersection of \mathbb{C}_+ or \mathbb{C}_- with the region bounded between the rays

$$\{\zeta : \arg(1 + \zeta) = \pm \frac{2}{3}\pi\} \quad \text{and} \quad \{\zeta : \arg(\zeta) = \pm \frac{2}{3}\pi\}.$$

Proving we have L^2 continuity here for $\beta \downarrow 0$ will imply the same for the original problem.

Similar to above, we now set

$$P_\beta(\zeta) = \begin{pmatrix} 1 & \frac{1}{2\pi i} \int_{-1}^\beta \frac{f(s)}{s-\zeta} ds \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & C_\beta(\zeta) \\ 0 & 1 \end{pmatrix}, \quad (7.6)$$

where

$$f(s) = 1 \text{ for } -1 < s < 0 \text{ and } f(s) = e^{-\frac{4}{3}s^{3/2}} \text{ for } 0 \leq s < \beta. \quad (7.7)$$

Again, the point is that $P_\beta(\zeta)$ satisfies the jump condition

$$(P_\beta)_+(\zeta) = (P_\beta)_-(\zeta) \begin{pmatrix} 1 & f(\zeta) \\ 0 & 1 \end{pmatrix}, \quad \zeta \in (-1, \beta).$$

Conjugating out by $P_\beta(\zeta)$ inside a disk $D_{1/2} = \{\zeta : |z - 1/2| < 1/2\}$ has three affects. First, a new jump of $P_\beta^{-1}(\zeta)$ is produced along $\partial D_{1/2}$. Second, the jump which $\widehat{M}(\zeta)$ has across $[0, \beta]$ is eliminated. Third, the jump across $[-1/2, 0]$ now reads

$$\begin{aligned} & \begin{pmatrix} 1 & (C_\beta)_-(\zeta) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{4}{3}\zeta_+^{3/2}} & 1 \\ 0 & e^{-\frac{4}{3}\zeta_-^{3/2}} \end{pmatrix} \begin{pmatrix} 1 & -(C_\beta)_+(\zeta) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{4}{3}\zeta_-^{3/2}} & 1 + (C_\beta)_-(\zeta)e^{\frac{4}{3}\zeta_-^{3/2}} - (C_\beta)_+(\zeta)e^{-\frac{4}{3}\zeta_+^{3/2}} \\ 0 & e^{\frac{4}{3}\zeta_-^{3/2}} \end{pmatrix}. \end{aligned} \quad (7.8)$$

We already understand that the jump $P_\beta^{-1}(\zeta)$ is continuous in $L^\infty \cap L^2$ of $\partial D_{1/2}$. To check that the jump (7.8) satisfies the like conditions over $-1/2 \leq \zeta \leq 0$, note that it is only the $(2, 1)$ -entry which requires investigation and that term (neglecting the constant 1) may be rewritten as in,

$$\begin{aligned} & (C_\beta)_-(\zeta)e^{\frac{4}{3}\zeta_-^{3/2}} - (C_\beta)_+(\zeta)e^{-\frac{4}{3}\zeta_+^{3/2}} \\ &= (C_\beta)_-(\zeta) - (C_\beta)_+(\zeta) + (C_\beta)_-(\zeta) \left(e^{\frac{4}{3}\zeta_-^{3/2}} - 1 \right) - (C_\beta)_+(\zeta) \left(e^{-\frac{4}{3}\zeta_+^{3/2}} - 1 \right) \\ &= -1 + (C_\beta)_-(\zeta) \left(e^{\frac{4}{3}\zeta_-^{3/2}} - 1 \right) - (C_\beta)_+(\zeta) \left(e^{-\frac{4}{3}\zeta_+^{3/2}} - 1 \right). \end{aligned} \quad (7.9)$$

The continuity in $L^2[-1/2, 0]$ as $\beta \downarrow 0$ follows by a computation similar to (7.6) and the boundedness of $e^{\pm \frac{4}{3}\zeta_\pm^{3/2}} - 1$. As for the continuity in L^∞ recall that the maps C_\pm maintain Hölder continuity, so there is no problem for ζ in the interior of $[-1/2, 0]$. The potential issue of the logarithmic

singularity of $(C_\beta)_\pm(0)$ as $\beta \downarrow 0$ is countered by the fact that $e^{\mp \frac{4}{3}\zeta^{3/2}} - 1$ vanish to higher order at the origin.

The criteria (7.1) has thus been checked for the new *RHP* created by the conjugation by $P_\beta(\zeta)$ defined in (7.6) and (7.7) within a neighborhood $D_{1/2}$ of the origin. It remains to invert this move and show that L^2 -continuity at $\alpha = 0$ of \widehat{M}_\pm (and so M_\pm^\pm) follows suit. However, the needed argument is identical to that given above in (7.5) and surrounding discussion. \square

Proof of Lemma 7.2 for RHP^\leftarrow . The verification of the conditions in this case is straightforward on account of the contour Σ^\leftarrow being independent of α from the start. For any positive α and β , the difference of V_β^\leftarrow and V_α^\leftarrow of course vanishes on R_- , while on the lines $\gamma_\pm = \{\zeta : \arg(\zeta) = \pm \frac{2}{3}\pi\}$,

$$(V_\beta^\leftarrow - V_\alpha^\leftarrow)(\zeta) = \begin{pmatrix} 0 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}}(e^{2\beta\zeta^{1/2}} - e^{2\alpha\zeta^{1/2}}) & 0 \end{pmatrix}.$$

Due to the decay of $e^{4/3\zeta^{3/2}}$ along γ^\pm it is plain that

$$\|e^{\frac{4}{3}\zeta^{3/2}}(e^{2\beta\zeta^{1/2}} - e^{2\alpha\zeta^{1/2}})\|_{L^\infty(\gamma^\pm) \cap L^2(\gamma^\pm)} \rightarrow 0,$$

as $\beta \rightarrow \alpha$, the case of $\alpha = 0$ and $\beta \downarrow 0$ being no different. \square

7.2 Asymptotics as $\alpha \rightarrow \pm \infty$

As $\alpha \rightarrow +\infty$, it is intuitive that the (unique) solution of *RHP* $_\rightarrow$ should converge to the solution $P_A(\zeta)$ of the *RHP* defined by the jump conditions,

$$\begin{aligned} (P_A)_+(\zeta) &= (P_A)_-(\zeta) \begin{pmatrix} 1 & e^{-\frac{4}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix}, & \zeta \in R_+, \\ (P_A)_+(\zeta) &= (P_A)_-(\zeta) \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}, & \arg \zeta = \pm \frac{2}{3}\pi, \\ (P_A)_+(\zeta) &= P_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \zeta \in R_-, \end{aligned} \tag{7.10}$$

with $P(\zeta)$ having the same asymptotics as $M^\rightarrow(\zeta)$ as $\zeta \rightarrow \infty$. As is well known, $P(\zeta)$ is given explicitly in terms of the Airy function $\text{Ai}(\zeta)$ and its derivative. In particular, with $\omega = e^{\frac{2}{3}\pi i}$, let

$$\begin{aligned} P(\zeta) &= \begin{pmatrix} \text{Ai}(\zeta) & \text{Ai}(\omega^2\zeta) \\ \text{Ai}'(\zeta) & \text{Ai}'(\omega^2\zeta) \end{pmatrix}, & \zeta \in \mathbb{C}_+, \\ P(\zeta) &= \begin{pmatrix} \text{Ai}(\zeta) & -\omega^2\text{Ai}(\omega^2\zeta) \\ \text{Ai}'(\zeta) & -\text{Ai}'(\omega^2\zeta) \end{pmatrix}, & \zeta \in \mathbb{C}_-, \end{aligned} \tag{7.11}$$

and $\Upsilon(\zeta) = \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2}} & 1 \end{pmatrix}$. Then,

$$\begin{aligned} P_A(\zeta) &= \sqrt{2\pi} e^{-\pi i/12} P(\zeta) e^{(\frac{2}{3}\zeta^{3/2} - \frac{\pi i}{6})\sigma_3}, & -\frac{2}{3}\pi < \arg \zeta < \frac{2}{3}\pi, \\ P_A(\zeta) &= \sqrt{2\pi} e^{-\pi i/12} P(\zeta) e^{(\frac{2}{3}\zeta^{3/2} - \frac{\pi i}{6})\sigma_3} \Upsilon(\zeta)^{-1}, & \frac{2}{3}\pi < \arg \zeta < \pi, \\ P_A(\zeta) &= \sqrt{2\pi} e^{-\pi i/12} P(\zeta) e^{(\frac{2}{3}\zeta^{3/2} - \frac{\pi i}{6})\sigma_3} \Upsilon(\zeta), & -\pi < \arg \zeta < -\frac{2}{3}\pi \end{aligned} \tag{7.12}$$

We have the following.

Lemma 7.4. *As $\alpha \rightarrow +\infty$,*

$$M^\rightarrow(\zeta)(P_A)^{-1}(\zeta) = \left(I + \mathcal{O}(e^{-\frac{2}{3}\alpha^{3/2}})\right), \quad (7.13)$$

uniformly for ζ supported away from $\mathbb{C} \setminus [\alpha, \infty)$.

If instead $\alpha \rightarrow -\infty$, one takes advantage of two facts. First, the jump for RHP^\leftarrow along $\arg \zeta = \pm \frac{2}{3}\pi$ satisfies

$$\begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{3/2} + 2\alpha\zeta^{1/2}} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-2|\alpha|\zeta^{1/2}(1+o(1))} & 1 \end{pmatrix}, \text{ for } \zeta = o(|\alpha|),$$

and, second, the unique solution of the RHP : $P_B(\zeta)$ analytic in $\mathbb{C} \setminus \Sigma^\rightarrow$,

$$\begin{aligned} (P_B)_+(\zeta) &= (P_B)_-(\zeta) \begin{pmatrix} 1 & 0 \\ e^{-2\zeta^{1/2}} & 1 \end{pmatrix}, \quad \zeta \in \gamma^\pm \\ (P_B)_+(\zeta) &= (P_B)_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \zeta \in (-\infty, 0), \end{aligned} \quad (7.14)$$

and

$$P_B(\zeta) = \zeta^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left(I + \mathcal{O}(|\zeta|^{-1/2})\right), \quad \zeta \rightarrow \infty, \quad (7.15)$$

is known explicitly in terms of Hankel functions. Here γ^\pm are any rays (eventually straight) rays extending above and below the negative real axis as in Figure 6. We have in fact already seen the solution in part. Set $Q(\zeta)$ to be as defined in (6.11), (6.12) and (6.13) but in regions I , II , and III respectively (see again Figure 6). Then,

$$P_B(\zeta) = \sqrt{2\pi} Q(\zeta) e^{-\zeta^{1/2}\sigma_3}. \quad (7.16)$$

That (7.16) satisfies the jumps (7.14) is immediate from the jump relations for $Q(\zeta)$. Note that replacing the straight lines $\arg \zeta = \pm 2\pi/3$ with $\zeta \in \gamma^\pm$ has no affect: $Q(\zeta)$ is analytic off \mathbb{R}^- and the jump contours may be deformed to accomodate this change. Lastly, the asymptotics (7.15) can be verified from substituting the formulas,

$$H_0^{(1)}(\zeta) = \sqrt{\frac{2}{\pi\zeta}} e^{i(\zeta - \frac{\pi}{4})} (1 + \mathcal{O}(\zeta^{-1/2})), \quad H_0^{(2)}(\zeta) = \sqrt{\frac{2}{\pi\zeta}} e^{-i(\zeta - \frac{\pi}{4})} (1 + \mathcal{O}(\zeta^{-1/2})),$$

for $\zeta \rightarrow \infty$ ([1], formulas 9.7.1 - 9.7.4) into the definition of $Q(\zeta)$.

The analogue of Lemma 7.4 can now be stated.

Lemma 7.5. *Set*

$$E_\alpha(\zeta) = \left(|\alpha| - \frac{2}{3}\zeta\right)^{\sigma_3/2} e^{-\frac{\pi i}{4}\sigma_3}, \quad \text{for } |\zeta| < \varepsilon|\alpha|, \quad (7.17)$$

with any $\varepsilon < 1$. Then γ^\pm in (7.14) may be chosen in such a way that

$$M^\leftarrow(\zeta) \left[E_\alpha(\zeta) P_B \left(\zeta \left(|\alpha| - \frac{2}{3}\zeta \right)^2 \right) \right]^{-1} = \left(I + \mathcal{O}(|\alpha|^{-1})\right), \quad \alpha \rightarrow -\infty, \quad (7.18)$$

uniformly on $|\zeta| < \varepsilon|\alpha|$.

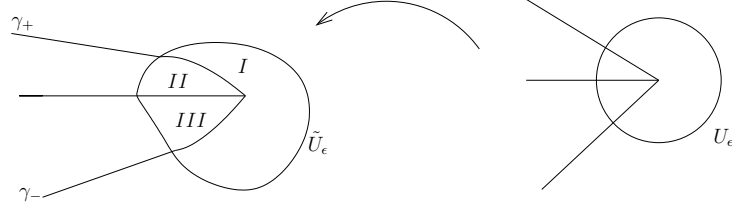


Figure 6: The contour for P_B and corresponding transformation.

From Lemmas 7.4 and 7.5, Corollary 1.4 is read off immediately from the explicit forms of P_A or $E_\alpha P_B$: we have for instance,

$$(M^\rightarrow)_{11}(\zeta) = \sqrt{2\pi} e^{-\pi i/4} \text{Ai}(\zeta) e^{\frac{2}{3}\zeta^{3/2}} (1 + \mathcal{O}(e^{-\frac{2}{3}\alpha^{3/2}}))$$

and

$$(M^\leftarrow)_{11}(\zeta) = \sqrt{2\pi} e^{-\pi i/4} (|\alpha| - \frac{2}{3}\zeta)^{1/2} I_0\left(\sqrt{\zeta}(|\alpha| - \frac{2}{3}\zeta)\right) e^{-\sqrt{\zeta}(|\alpha| - \frac{2}{3}\zeta)} (1 + \mathcal{O}(|\alpha|^{-1})).$$

Lemmas 7.4 and 7.5 themselves follow directly from checking condition (7.19) of the below proposition, the proof of which may be found in [12], Section 7.

Proposition 7.6. *If for a family of L^2 -solvable RHP's (Σ, v_n) there is the estimate*

$$\|v_n - I\|_{L^\infty(\Sigma) \cap L^2(\Sigma)} \leq \frac{C}{n}, \quad (7.19)$$

for a fixed constant C and all large n , then $\|C_{v_n}\|_{L^2(\Sigma) \rightarrow L^2(\Sigma)} = \mathcal{O}(\frac{1}{n})$, and the solutions satisfy $m_n(\zeta) = I + \mathcal{O}(\frac{1}{n})$ uniformly for ζ a positive distance from Σ .

The fact that (7.19) implies a like bound on the operator norm of C_{v_n} actually implies the existence of a full asymptotic expansion of $M^\rightarrow(\zeta)$ and $M^\leftarrow(\zeta)$ in powers of $e^{-\alpha^{3/2}}$ or α^{-1} with sectionally analytic coefficients. This is not pursued here.

Proof of Lemma 7.4. Define

$$R(\zeta) = M^\rightarrow(\zeta)(P_A)^{-1}(\zeta),$$

which solves the RHP:

$$\begin{aligned} R(\zeta) & \text{ analytic in } \mathbb{C} \setminus [\alpha, \infty) \\ R_+(\zeta) &= R_-(\zeta) \begin{pmatrix} (P_A)_-(\zeta) & \begin{pmatrix} 1 & -e^{-\frac{2}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix} \end{pmatrix} (P_A)_-^{-1}(\zeta), \quad \zeta \in [\alpha, \infty) \\ R(\zeta) &= I + \mathcal{O}(\frac{1}{\zeta}), \quad \zeta \rightarrow \infty. \end{aligned} \quad (7.20)$$

The jump matrix along $[\alpha, \infty)$ can be simplified as in

$$\begin{aligned} (P_A)_-(\zeta) \begin{pmatrix} 1 & -e^{-\frac{2}{3}\zeta^{3/2}} \\ 0 & 1 \end{pmatrix} (P_A)_-^{-1}(\zeta) &= I - e^{-\frac{\pi i}{3}} \begin{pmatrix} -\text{Ai}(\zeta)\text{Ai}'(\zeta) & \text{Ai}^2(\zeta) \\ (\text{Ai}'(\zeta))^2 & \text{Ai}(\zeta)\text{Ai}'(\zeta) \end{pmatrix} \\ &\equiv I - V_R(\zeta). \end{aligned}$$

Next, noting the asymptotics,

$$\begin{aligned} \text{Ai}(\zeta) &= \frac{z^{-1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}\zeta^{3/2}} (1 + \mathcal{O}(|\zeta|^{-3/2})), \\ \text{Ai}'(\zeta) &= \frac{-z^{-1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}\zeta^{3/2}} (1 + \mathcal{O}(|\zeta|^{-3/2})), \end{aligned} \quad \zeta \rightarrow \infty, \quad |\arg(\zeta)| \leq \frac{2}{3}\pi,$$

(see [1], p. 446), we have that both $\|V_R\|_{L^\infty[\alpha, \infty)}$ and $\|V_R\|_{L^2[\alpha, \infty)}$ are bounded by constant multiples of $e^{-\alpha^{3/2}}$, and the claim follows. \square

Proof of Lemma 7.5. First we consider the scaled *RHP* for

$$M^{(1)}(w) \equiv M^\leftarrow(|\alpha|w),$$

which has the jump conditions:

$$\begin{pmatrix} 1 & 0 \\ e^{-|\alpha|^{3/2}(2w^{1/2} - \frac{4}{3}w^{3/2})} & 1 \end{pmatrix}, \quad \arg w = \pm \frac{2}{3}\pi, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad w \in \mathbb{R}_-.$$

We will now extract a local parametrix in a neighborhood of $w = 0$. For $w \in U_\varepsilon = \{|w| < \varepsilon\}$ and $\varepsilon < 1$ define

$$\eta = \eta(w) = w(1 - \frac{2}{3}w)^2$$

Clearly, $\eta(w)$ takes U_ε in a one-to-one fashion onto a open neighborhood \tilde{U}_ε of $\nu = 0$, sending the negative real line to itself and the segments $\arg w = \pm 2\pi/3$ onto rays $\gamma^\pm \subset \tilde{U}_\varepsilon$ lying above and below the real axis. Extending γ^\pm to ∞ (smoothly) along straight lines outside of \tilde{U}_ε we have the jump relations

$$\begin{pmatrix} 1 & 0 \\ e^{-2|\alpha|^{3/2}\eta^{1/2}} & 1 \end{pmatrix}, \quad \eta \in \gamma^\pm \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \eta \in \mathbb{R}_-.$$

This identifies the choice of γ^\pm , and with this choice the above problem is solved by $P_B(|\alpha|^3\eta)$. What is the same, $P_B(|\alpha|^3\eta(w))$ satisfies the jump relations for $M^{(1)}(w)$ restricted to $|w| < \varepsilon$ (in which the upper and lower contours are pulled back to $\arg w = \pm 2\pi/3$).

Next we perform a second transformation, setting

$$M^{(2)}(w) = \begin{cases} M^{(1)}(w) \left(\mathbf{m}(|\alpha|w) \right)^{-1}, & |w| > \varepsilon, \\ M^{(1)}(w) \left(E_\alpha(|\alpha|w) P_B(|\alpha|^3\eta(w)) \right)^{-1}, & |w| < \varepsilon \end{cases}. \quad (7.21)$$

The definition of $E_\alpha(|\alpha|w)$ is given in (7.17). It is analytic in $|w| < \varepsilon$ and so $E_\alpha(|\alpha|w) P_B(|\alpha|^3\eta(w))$ also shares jump conditions with $M^{(1)}(w)$ in $|w| < \varepsilon$.

The point is that $M^{(2)}(w)$ satisfies a new *RHP* with jump contour consisting of three pieces: the rays $\Gamma^\pm \equiv \arg w = \pm 2\pi/3 \cap \{|w| > \varepsilon\}$ and the ∂U_ε the boundary of the disk of radius ε .

On either of the first set of contours, Γ^\pm , the jump matrix is bounded as in

$$\left| \mathbf{m}(|\alpha|w) \begin{pmatrix} 1 & 0 \\ e^{-|\alpha|^{3/2}(w - \frac{2}{3}w^{3/2})} & 0 \end{pmatrix} (\mathbf{m}(|\alpha|w))^{-1} \right| \leq I + e^{-\frac{2}{3}|\alpha|^3|w|^{3/2}} \begin{pmatrix} 1 & 1 \\ \alpha^2|z|^{1/2} & 1 \end{pmatrix}$$

which is to say it is $I + \mathcal{O}(e^{-C_\varepsilon|\alpha|^3})$ in $L^2 \cap L^\infty(\Gamma^\pm)$. On ∂U_ε the jump matrix is $E_\alpha(|\alpha|w) P_B(|\alpha|^3\eta(w)) (\mathfrak{m}(|\alpha|w))^{-1}$ and we compute: for $|w| = \varepsilon$ and $|\alpha| \gg 1$,

$$\begin{aligned} E_\alpha(|\alpha|w) P_B(|\alpha|^3\eta(w)) (\mathfrak{m}(|\alpha|w))^{-1} &= E_\alpha(|\alpha|w) \left(\frac{w}{\alpha^2\eta(w)} \right)^{\sigma_3/4} e^{\frac{\pi i}{4}\sigma_3} \left(I + \mathcal{O}(\alpha^{-1}) \right) \\ &= I + \mathcal{O}(\alpha^{-1}). \end{aligned}$$

It follows that

$$M^{(2)}(w) = I + \mathcal{O}(\alpha^{-1}), \text{ uniformly for } w \text{ supported away from } \partial U_\varepsilon \cup \Gamma^+ \cup \Gamma^-.$$

Undoing the transformations inside of U_ε establishes the claim for $M^\leftarrow(\zeta)$. \square

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